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# ON SOME CHARACTER RELATIONS OF SYMMETRIC GROUPS

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**1.** Let $n$ be a natural number and let

$$(1) \qquad n = \alpha_1 + \alpha_2 + \cdots\cdots + \alpha_h, \qquad \alpha_1 \geqq \alpha_2 \geqq \cdots\cdots \geqq \alpha_h > 0$$

be a partition $(\alpha_i)$ of $n$ into $h$ natural numbers $\alpha_i$. By a diagram $T = [\alpha_i]$ corresponding to this partition we mean an arrangement of $n$ nodes into $h$ rows consisting of $\alpha_1, \alpha_2, \cdots\cdots, \alpha_h$ nodes. The number $m(n)$ of distinct diagrams of $n$ nodes is equal to the number of irreducible representations of the symmetric group $\mathfrak{S}_n$. We set $m(0) = 1$. If $p$ is a fixed prime number, then the number $s(n)$ of diagrams without $p$-hook[1] is equal to the number of $p$-blocks of highest kind[2]. Let

$$(2) \qquad n = kp + r, \qquad 0 \leqq r < p.$$

By R. Brauer[3], the number of $p$-blocks of $\mathfrak{S}_n$ is equal to $\sum_{\lambda=0}^{k} s(n-\lambda p)$. Now we define $l(\lambda)$ and $l^*(\lambda)$ by

$$(3) \qquad l(\lambda) = \sum_{\lambda_1, \lambda_2, \cdots\cdots, \lambda_p} m(\lambda_1)m(\lambda_2)\cdots\cdots m(\lambda_p) \qquad (\textstyle\sum \lambda_i = \lambda,\ 0 \leqq \lambda_i \leqq \lambda)$$

$$(4) \qquad l^*(\lambda) = \sum_{\nu_1, \nu_2, \cdots\cdots, \nu_{p-1}} m(\nu_1)m(\nu_2)\cdots\cdots m(\nu_{p-1}) \qquad (\textstyle\sum \nu_i = \lambda,\ 0 \leqq \nu_i \leqq \lambda).$$

Let $T_0$ be a diagram of $\mathfrak{S}_{n-\lambda p}$ without $p$-hook. Then $T_0$ determines uniquely a $p$-block $B_\sigma$ of $\mathfrak{S}_n$, and the number of ordinary irreducible characters in $B_\sigma$ is given by $l(\lambda)$[4]. Hence we have

$$(5) \qquad m(n) = \sum_{\lambda=0}^{k} s(n-\lambda p)l(\lambda).$$

**Lemma 1.** $\qquad l(\lambda) - l^*(\lambda) = \sum_{\beta=1}^{\lambda} l^*(\lambda-\beta)m(\beta).$

1) For the notion of hooks, see T. Nakayama, *On some modular properties of the irreducible representations of symmetric groups* I. II, Jap. J. Math. 17 (1941): we refer to these papers as NI and NII.

2) See NII. p. 413.

3) R. Brauer, *On a conjecture by Nakayama*, Trans. Roy. Soc. Canada, 41 (1947).

4) T. Nakayama and M. Osima, *Note on blocks of symmetric groups*, Nagoya Math. J. 2 (1951).

*Proof.* From our definition

$$l(\lambda) = \sum_{\lambda p=0}^{\lambda} l^*(\lambda - \lambda_p)m(\lambda_p) = l^*(\lambda) + \sum_{\lambda p=1}^{\lambda} l^*(\lambda - \lambda_p)m(\lambda_p).$$

Let $C_1, C_2, \cdots\cdots, C_{m(n)}$ be the classes of conjugate elements in $\mathfrak{S}_n$. If $C_\nu$ contains an element $G$ such that $G$ is a permutation composed of cycles of lengths $\alpha_1, \alpha_2, \cdots\cdots, \alpha_h$ ($\alpha_1 \geqq \alpha_2 \geqq \cdots\cdots \geqq \alpha_h > 0$), then $C_\nu$ is characterized by a partition $(\alpha_i)$. Hence we denote $C_\nu$ by $C(\alpha_i)$.

**Lemma 2.** *The number of classes $C(\alpha_i)$ in $\mathfrak{S}_{kp}$ such that $\alpha_i = \beta_i p$ ($i = 1, 2, \cdots\cdots$) is equal to $m(k)$.*

*Proof.* Every $C(\beta_i p)$ determines uniquely a class $C(\beta_i)$ in $\mathfrak{S}_k$, and conversely.

A $p$-regular element of $\mathfrak{S}_n$ is an element whose order is prime to $p$; the other elements are said to be $p$-singular. Similarly, we denote the classes of conjugate elements as $p$-regular or $p$-singular according as the elements of the classes are $p$-regular or $p$-singular. Let us denote by $m^*(n)$ the number of $p$-regular classes in $\mathfrak{S}_n$. Then we have

**Lemma 3.** $$m(n) - m^*(n) = \sum_{\beta=1}^{k} m^*(n - \beta p)m(\beta).$$

*Proof.* If $C(\alpha_i)$ is a $p$-singular class, then at least one $\alpha_i$ is divisible by $p$. Let

$$\alpha_{\lambda(1)} = \beta_1 p, \quad \alpha_{\lambda(2)} = \beta_2 p, \cdots\cdots, \alpha_{\lambda(t)} = \beta_t p$$

$\beta_1 \geqq \beta_2 \geqq \cdots\cdots \geqq \beta_t > 0$ and the remaining $\alpha_i$ be prime to $p$. Such $\alpha_i$ we denote by $\gamma_1, \gamma_2, \cdots\cdots, \gamma_{h-t}$:

$$\gamma_1 \geqq \gamma_2 \geqq \cdots\cdots \geqq \gamma_{h-t} > 0, \qquad (\gamma_j, p) = 1.$$

If $\sum \beta_i = \beta$, then $C(\alpha_i)$ determines uniquely $C(\beta_i)$ in $\mathfrak{S}_\beta$ and $p$-regular $C(\gamma_j)$ in $\mathfrak{S}_{n-\beta p}$. Since the converse is also valid, we obtain our assertion by Lemma 2.

**Theorem 1.** *Let $m^*(n)$ be the number of $p$-regular classes in $\mathfrak{S}_n$. Then $m^*(n) = \sum_{\lambda=0}^{k} s(n - \lambda p)l^*(\lambda)$.*

*Proof.* Our assertion is evidently valid when $k = 0$, that is, $n < p$. Let $k > 0$ and assume that the theorem is true for $\mathfrak{S}_{n-\beta p}$ ($\beta = 1, 2, \cdots\cdots, k$). Then we have

$$m^*(n-\beta p)=\sum_{\sigma=0}^{k-\beta}s(n-(\beta+\sigma)p)l^*(\sigma)=\sum_{\lambda=\beta}^{k}s(n-\lambda p)l^*(\lambda-\beta).$$

Hence it follows from Lemma 3 that

$$\begin{aligned}m(n)-m^*(n)&=\sum_{\beta=1}^{k}(\sum_{\lambda=\beta}^{k}s(n-\lambda p)l^*(\lambda-\beta))m(\beta)\\&=\sum_{\lambda=1}^{k}(\sum_{\beta=1}^{\lambda}l^*(\lambda-\beta)m(\beta))s(n-\lambda p)\\&=\sum_{\lambda=1}^{k}(l(\lambda)-l^*(\lambda))s(n-\lambda p)\\&=m(n)-\sum_{\lambda=1}^{k}s(n-\lambda p)l^*(\lambda).\end{aligned}$$

Whence we have $m^*(n)=\sum_{\lambda=1}^{k}s(n-\lambda p)l^*(\lambda)$.

**2.** Let $\chi_1, \chi_2, \cdots\cdots, \chi_{m(n)}$ be the distinct ordinary irreducible characters of $\mathfrak{S}_n$. Let us denote by $C(G)$ a class of conjugate elements which contains an element $G$. Since $G^{-1}\in C(G)$, we have

$$\chi_i(G)=\chi_i(G^{-1}) \qquad i=1,2,\cdots\cdots,m(n). \tag{6}$$

From the orthogonality relations for ordinary group characters, we have

$$\begin{aligned}\sum_{i=1}^{m(n)}\chi_i(G_\mu)\chi_i(G_\nu^{-1})&=\sum_{i=1}^{m(n)}\chi_i(G_\mu)\chi_i(G_\nu)\\&=\begin{cases}n(G_\mu) & \text{for } C(G_\mu)=C(G_\nu)\\ 0 & \text{for } C(G_\mu)\neq C(G_\nu)\end{cases}\end{aligned} \tag{7}$$

where $n(G_\mu)$ is the order of the normalizer $N(G_\mu)$. If $V$ is any $p$-regular element of $\mathfrak{S}_n$, then among $m(n)$ characters $\chi_1(V), \chi_2(V), \cdots\cdots, \chi_{m(n)}(V)$, there exist $m^*(n)$ linearly independent $\chi_j(V)$. In the following we shall determine all linear relations between $\chi_1(V), \chi_2(V), \cdots\cdots, \chi_{m(n)}(V)$ by Murnaghan's recurrence rule[1].

**Murnaghan's recurrence rule.** *Let $H_1, H_2, \cdots\cdots$ be the totality of g-hooks in the diagram $T$, and let $r_\nu$ be the height of $H_\nu$. If $Q$ is an element of $\mathfrak{S}_n$ containing a g-cycle and if $\bar{Q}$ is the permutation of $n-g$ letters obtained from $Q$ by removing this cycle, then*

$$\chi(T:Q)=(-1)^{r_1-1}\chi(T-H_1:\bar{Q})+(-1)^{r_2-1}\chi(T-H_2:\bar{Q})+\cdots\cdots$$

1) F. D. Murnaghan, *On the representations of the symmetric group*, Amer. J. Math. 59 (1937). Cf. also NI, Appendix.

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*where* $\chi(T)$, $\chi(T-H_\nu)$ *denote the characters belonging to the diagrams* $T$, $T-H_\nu$. *If* $T$ *possesses no g-hook, then* $\chi(T: Q)=0$.

Let us denote by $Q$ the elements of $\mathfrak{S}_n$ containing at least one $g$-cycle. Then there exist $m(n-g)$ classes $C(Q_1)$, $C(Q_2)$, ......, $C(Q_{m(n-g)})$ which contain the elements $Q$. If $\bar{Q}_\nu$ is the permutation of $n-g$ letters obtained from $Q_\nu$ by removing a $g$-cycle, then $C(\bar{Q}_1)$, $C(\bar{Q}_2)$, ......, $C(\bar{Q}_{m(n-g)})$ are all the classes of conjugate elements in $\mathfrak{S}_{n-g}$. From (6), we have

$$(8)\qquad \begin{cases} \sum_{i=1}^{m(n)} \chi_i(U)\chi_i(Q_\nu) = 0 \\ \sum_{i=1}^{m(n)} \chi_i(Q_\mu)\chi_i(Q_\nu) = n(Q_\mu)\delta_{\mu\nu} \end{cases}$$

where $U$ is any element of $\mathfrak{S}_n$ without $g$-cycle. Applying the recurrence rule to $\chi_i(Q_\nu)$ in (8), we obtain

$$(9)\qquad \begin{cases} \sum_{j=1}^{m(n-g)} R_j(\chi_i(U))\chi_j^*(\bar{Q}_\nu) = 0 \\ \sum_{j=1}^{m(n-g)} R_j(\chi_i(Q_\mu))\chi_j^*(\bar{Q}_\nu) = n(Q_\mu)\delta_{\mu\nu} \end{cases}$$

where $\chi_j^*$ ($j=1,2,\cdots\cdots,m(n-g)$) are the irreducible characters of $\mathfrak{S}_{n-g}$ and $R_j(\chi_i(G))$ for any $G\in\mathfrak{S}_n$ is a linear combination of $\chi_1(G)$, $\chi_2(G)$, ......, $\chi_{m(n)}(G)$. Since $\chi_1^*$, $\chi_2^*$, ......, $\chi_{m(n-g)}^*$ are linearly independent, we have from the first formula (9)

$$(10)\qquad R_j(\chi_i(U)) = 0 \qquad j=1,2,\cdots\cdots,m(n-g)$$

for all elements $U$ without $g$-cycle.

**Lemma 4.** *Let* $T^*$ *be a diagram of* $\mathfrak{S}_{n-g}$. *If* $T^*$ *contains* $\rho(r)$ *g-hooks of the same height r, then we can obtain* $\rho(r)+1$ *distinct diagrams of* $\mathfrak{S}_n$ *by adjoining a g-hook of the height r to* $T^*$.

*Proof.* When $\rho(r)=0$, our assertion is valid by T. Nakayama[1]. Hence, by induction with respect to $\rho(r)$, we can show that our assertion is true for any $\rho(r)$.

Let $T_j^*$ be the diagram of $\mathfrak{S}_{n-g}$ corresponding to $\chi_j^*$, and let

$$T_{j,1}^{(r)},\ T_{j,2}^{(r)},\ \cdots\cdots,\ T_{j,\rho(r)+1}^{(r)}$$

be the diagram of $\mathfrak{S}_n$ obtained from $T_j^*$ by adjoining a $g$-hook of the

1) See XII. p. 414.

height $r$. If we denote by $\chi_{j,\sigma}^{(r)}$ the irreducible character belonging to $T_{j,\sigma}^{(r)}$, then we can see that

(11) $$R_j(\chi_i(G)) = \sum_{r=1}^{g}\sum_{\sigma=1}^{\rho(r)+1}(-1)^{r-1}\chi_{j,\sigma}^{(r)}(G) \qquad \text{(for all } G \in \mathfrak{S}_n).$$

**Theorem 2.** *$R_1(\chi_i(G))$, $R_2(\chi_i(G))$, ……, $R_{m(n-g)}(\chi_i(G))$ are linearly independent.*

*Proof.* If we set

$$M = R_j(\chi_i(Q_\mu)), \qquad Z = (\chi_j^*(\bar{Q}_\mu))$$

($j$ row index, $\mu$ column index: $j, \mu = 1, 2, \cdots\cdots, m(n-g)$), then the second formula (9) becomes

$$Z'M = (n(Q_\mu)\delta_{\mu\nu}) = D.$$

Since $D$ is non-singular, we have $|M| \neq 0$. Hence $R_1(\chi_i(Q_\mu))$, $R_2(\chi_i(Q_\mu))$, ……, $R_{m(n-g)}(\chi_i(Q_\mu))$ ($\mu = 1, 2, \cdots\cdots, m(n-g)$) are linearly independent. This fact shows that the theorem is valid.

If we put, in particular, $g = \lambda p$ ($\lambda = 1, 2, \cdots\cdots, k$) in (10), then we obtain

(12) $$R_j^{(\lambda)}(\chi_i(V)) = 0 \qquad j = 1, 2, \cdots\cdots, m(n-\lambda p), \quad \lambda = 1, 2, \cdots\cdots, k.$$

where $V$ is any $p$-regular element of $\mathfrak{S}_n$.

**Lemma 5.** $m(n) - m^*(n) \leqq \sum_{\lambda=1}^{k} m(n-\lambda p)$.

For the sake of simplicity, we set $u = m(n) - m^*(n)$ and $v = \sum_{\lambda=1}^{k} m(n-\lambda p)$. Let us denote by

(13) $$C(P_1^{(\lambda)}),\ C(P_2^{(\lambda)}),\ \cdots\cdots,\ C(P_{d(\lambda)}^{(\lambda)})$$

the $p$-singular classes in $\mathfrak{S}_n$ such that $P_\mu^{(\lambda)}$ contains a $\lambda p$-cycle but does not contain a $\lambda' p$-cycle ($\lambda < \lambda'$). Then $C(P_\mu^{(\lambda)})$ ($\mu = 1, 2, \cdots\cdots, d(\lambda)$, $\lambda = 1, 2, \cdots\cdots, k$) give all the $p$-singular classes in $\mathfrak{S}_n$. Hence

(14) $$u = \sum_{\lambda=1}^{k} d(\lambda).$$

Let $\bar{P}_\mu^{(\lambda)}$ be an element of $\mathfrak{S}_{n-\lambda p}$, obtained from $P_\mu^{(\lambda)}$ by removing a $\lambda p$-cycle. Then, similarly as (9), we have

1) By a matrix of type $(a, b)$ we understand a matrix with $a$ rows and $b$ columns.

$$(15)\qquad \begin{cases} \sum_{j=1}^{m(n-\lambda p)} R_j^{(\lambda)}(\chi_i(P_\mu^{(\lambda)}))\chi_j^{(\lambda)}(\bar{P}_\nu^{(\lambda)}) = n(P_\mu^{(\lambda)})\delta_{\mu\nu} \\ \sum_{j=1}^{m(n-\lambda p)} R_j^{(\lambda)}(\chi_i(P_\mu^{(\kappa)}))\chi_j^{(\lambda)}(\bar{P}_\nu^{(\lambda)}) = 0 \qquad (\kappa \neq \lambda). \end{cases}$$

If we set

$$M_{\lambda\kappa} = (R_j^{(\lambda)}(\chi_i(P_\nu^{(\kappa)})), \qquad Z_\lambda = (\chi_j^{(\lambda)}(\bar{P}_\mu^{(\lambda)}))'$$

($j$ row index, $\mu$, $\nu$ column index: $j = 1, 2, \cdots\cdots, m(n-\lambda p)$, $\nu = 1, 2, \cdots\cdots, d(\kappa)$, $\mu = 1, 2, \cdots\cdots, d(\lambda)$), then (15) becomes

$$(16)\qquad \begin{cases} Z_\lambda' M_{\lambda\lambda} = (n(P_\mu^{(\lambda)})\delta_{\mu\nu}) = D_\lambda \\ Z_\lambda' M_{\lambda\kappa} = 0. \end{cases}$$

Hence we have

$$\begin{pmatrix} Z_1' & & & \\ & Z_2' & & \\ & & \ddots & \\ & & & Z_k' \end{pmatrix} \begin{pmatrix} M_{11} & M_{12} & \cdots\cdots & M_{1k} \\ M_{21} & M_{22} & \cdots\cdots & M_{2k} \\ \cdots & \cdots & \cdots & \cdots \\ M_{k1} & M_{k2} & \cdots\cdots & M_{kk} \end{pmatrix} = \begin{pmatrix} D_1 & & & \\ & D_2 & & \\ & & \ddots & \\ & & & D_k \end{pmatrix}.$$

Since $D_\lambda$ ($\lambda = 1, 2, \cdots\cdots, k$) are non-singular, the matrix $(M_{\kappa\lambda})$ ($\kappa, \lambda = 1, 2, \cdots\cdots, k$) which is of type $(v, u)$ has a rank $u = \sum_{\lambda=1}^{k} d(\lambda)$. This implies that there exist $u$ linearly independent $R_k^{(\lambda)}(\chi_i(P))$ among $v$ $R_j^{(\lambda)}(\chi_i(P))$ where $P$ is any $p$-singular element of $\mathfrak{S}_n$. *This fact, combined with* (12), *shows that if* $R(\chi_i(V)) = \sum a_i\chi_i(V) = 0$ *for all $p$-regular elements $V$, then* $R(\chi_i(G))$ (*for any* $G \in \mathfrak{S}_n$) *is a linear combination of* $R_j^{(\lambda)}(\chi_i(G))$.

The relations (12) seem to be useful to determine the irreducible modular characters of $\mathfrak{S}_n$, but we have only succeeded to determine the characters belonging to the $p$-blocks of next-highest kind.

In the forthcoming paper, we shall study the properties of $R_j^{(\lambda)}(\chi_i(G))$ in detail.

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