

# *Mathematical Journal of Okayama University*

---

*Volume 1, Issue 1*

1952

*Article 2*

MARCH 1952

---

## On the representations of groups of finite order

Masaru Osima\*

\*

Copyright ©1952 by the authors. *Mathematical Journal of Okayama University* is produced by  
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

## ON THE REPRESENTATIONS OF GROUPS OF FINITE ORDER

MASARU OSIMA

### Introduction.

The representations of a group  $\mathfrak{G}$  of finite order  $g$  were first studied by G. Frobenius<sup>1)</sup> in his theory of group characters. The coefficients of the linear transformations are taken as complex numbers, but we may take them as the elements of an algebraically closed field of characteristic 0. Recently the modular representations of  $\mathfrak{G}$  (i.e. representations of  $\mathfrak{G}$  by matrices with coefficients in a modular field) which were first treated by L. E. Dickson<sup>2)</sup>, has been studied by R. Brauer and C. Nesbitt jointly and very interesting results have been obtained<sup>3)</sup>. In the present paper, we shall give a new method to the theory of group representations which enables us in particular to prove the orthogonality relations for group characters in a quite natural way.

In Part I, we study the properties of the regular representations of algebras. Let  $A$  be an algebra with unit element. Let  $A'$  be an algebra anti-isomorphic to  $A$  and  $a \rightarrow a'$  an anti-isomorphism between  $A$  and  $A'$ . If we denote by  $S(a)$  and  $R(a)$  the left and the right regular representations of  $A$ , then  $a \times b' \rightarrow S(a)R'(b)$  is a representation of the direct product  $A \times A'$  where  $R'(b)$  is the transpose of  $R(b)$ . We can derive the properties of the regular representations of  $A$  by studying the structure of the representation  $S(a)R'(b)$  of  $A \times A'$ . Theorem 1 and Theorem 2 play a principal role in our theory. Applying Theorem 1 to the group ring of  $\mathfrak{G}$  we can obtain the orthogonality relations for group characters. The relations for the induced characters of  $\mathfrak{G}$  are derived from Theorem 2. In Parts II and III, we study the ordinary representations and the modular representations of  $\mathfrak{G}$  respectively. In particular, we can obtain a simple proof of the

---

1) All of Frobenius' papers were published in the *Sitzungsber. Preuss. Akad.* A complete list of titles is to be found in Speiser (18). Three treatments of the theory were given by Burnside (7), Schur (16) and Noether (14). Cf. also the accounts in Dickson (10), Speiser (18) and Waerden (20).

2) Dickson (8), (9).

3) Brauer (2). Brauer-Nesbitt (4), (6).

fundamental relation between the Cartan invariants and the decomposition numbers of  $\mathfrak{G}$ <sup>1)</sup>. The last Part deals with the representations of  $\mathfrak{G}$  by collineations.

### I. Regular representations of algebras.<sup>2)</sup>

1. Let  $A$  be an (associative) algebra with unit element 1 over an algebraically closed field  $K$ , and  $N$  be the radical of  $A$ . Let

$$\bar{A} = A/N = \bar{A}_1 + \bar{A}_2 + \dots + \bar{A}_n$$

be a decomposition of residue class algebra  $\bar{A} = A/N$  into a direct sum of simple two-sided ideals  $\bar{A}_\lambda$ . Denote by  $\bar{E}_1, \bar{E}_2, \dots, \bar{E}_n$  the unit elements of  $\bar{A}_1, \bar{A}_2, \dots, \bar{A}_n$ . Each  $\bar{E}_\lambda$  can be decomposed into a sum of mutually orthogonal idempotent elements  $\bar{e}_{\lambda,1}, \bar{e}_{\lambda,2}, \dots, \bar{e}_{\lambda,f(\lambda)}$  such that left ideals  $\bar{A}\bar{e}_{\lambda,i}$  as well as right ideals  $\bar{e}_{\lambda,i}\bar{A}$  are simple. There exist mutually orthogonal idempotent elements  $e_{\lambda,i}$  in  $A$  such that  $e_{\lambda,i} \pmod{N} = \bar{e}_{\lambda,i}$  ( $\lambda = 1, 2, \dots, n; i = 1, 2, \dots, f(\lambda)$ ). If we put  $E_\lambda = e_{\lambda,1} + e_{\lambda,2} + \dots + e_{\lambda,f(\lambda)}$ , then  $E_1 + E_2 + \dots + E_n = 1$ .  $A$  is a direct sum:

$$A = Ae_{1,1} + \dots + Ae_{1,f(1)} + Ae_{2,1} + \dots + Ae_{n,f(n)} \\ [A = e_{1,1}A + \dots + e_{1,f(1)}A + e_{2,1}A + \dots + e_{n,f(n)}A].$$

The idempotent elements  $e_{\lambda,i}$  are primitive and the left ideals  $Ae_{\lambda,i}$  as well as the right ideals  $e_{\lambda,i}A$  are directly indecomposable. Further  $Ae_{\lambda,i} [e_{\lambda,i}A]$  with one and the same first suffix  $\lambda$ , and only those are (operator-) isomorphic to each other:

$$Ae_{\lambda,1} \cong Ae_{\lambda,2} \cong \dots \cong Ae_{\lambda,f(\lambda)} [e_{\lambda,1}A \cong e_{\lambda,2}A \cong \dots \cong e_{\lambda,f(\lambda)}A].$$

For the sake of simplicity, let us denote one of  $e_{\lambda,i}$  ( $i = 1, 2, \dots, f(\lambda)$ ), say  $e_{\lambda,1}$ , by  $e_\lambda$ . Incidentally we denote  $\bar{e}_{\lambda,1}$  by  $\bar{e}_\lambda$ . Let  $U_\lambda$  and  $V_\lambda$  be the indecomposable representations of  $A$  belonging to the left ideal  $Ae_\lambda$  and the right ideal  $e_\lambda A$  respectively. Then

$$U_\lambda \cong \begin{pmatrix} F_\lambda & \\ * & M_\lambda \end{pmatrix}, \quad V_\lambda \cong \begin{pmatrix} F_\lambda & * \\ & N_\lambda \end{pmatrix}$$

where  $M_\lambda$  and  $N_\lambda$  are (reducible or irreducible) representations of  $A$

1) See Brauer-Nesbitt (4), Nakayama (11) and Brauer (3).

2) Cf. Brauer-Nesbitt (5), Nakayama (11) and Nesbitt (13).

and  $F_\lambda$  is the irreducible representation of  $A$  belonging to  $\bar{A}e_\lambda[\bar{e}_\lambda\bar{A}]$ . All these are well known.

Let  $m_1, m_2, \dots, m_t$  be a basis of  $A$ . For every  $a$  in  $A$ , we have equations

$$(1.1) \quad \begin{aligned} am_\lambda &= \sum_{\kappa} s_{\kappa\lambda} m_\kappa \\ m_\lambda a &= \sum_{\kappa} r_{\kappa\lambda} m_\kappa \end{aligned}$$

where the coefficients  $s_{\kappa\lambda}$  and  $r_{\kappa\lambda}$  lie in  $K$ . We then obtain two representations of  $A$  by associating the matrices  $S(a) = (s_{\kappa\lambda})$ ,  $R(a) = (r_{\kappa\lambda})$  with  $a$ . These representations  $a \rightarrow S(a)$  and  $a \rightarrow R(a)$  are called the left and the right regular representations of  $A$  respectively. Since equations (1.1) become in matrix form

$$(1.2) \quad \begin{aligned} a(m_1 m_2 \dots m_t) &= (m_1 m_2 \dots m_t) S(a) \\ (m_1 m_2 \dots m_t) a &= (m_1 m_2 \dots m_t) R'(a) \end{aligned}$$

we have

$$(1.3) \quad a(m_1 m_2 \dots m_t) b = (m_1 m_2 \dots m_t) S(a) R'(b).$$

Let  $A'$  be an algebra anti-isomorphic to  $A$  and  $a \rightarrow a'$  an anti-isomorphism between  $A$  and  $A'$ . Then  $a' \rightarrow R'(a)$  is the left regular representation of  $A'$ . Since  $S(a)R'(b) = R'(b)S(a)$  for any  $a, b \in A$ ,  $a \times b \rightarrow S(a)R'(b)$  is a representation of the direct product  $A \times A'$ . (1.3) shows that the representation  $S(a)R'(b)$  of  $A \times A'$  belongs to the  $A$ -two-sided module  $A$ . Since  $a' \rightarrow F'_\lambda(a)$  ( $\lambda = 1, 2, \dots, n$ ) are the irreducible representations of  $A'$ , the distinct irreducible representations of  $A \times A'$  are given by  $F_\kappa(a) \times F'_\lambda(b)$  ( $\kappa, \lambda = 1, 2, \dots, n$ ) according to our assumption concerning  $K$ . Let  $c_{\kappa\lambda}$  denote the multiplicity of  $F_\kappa(a) \times F'_\lambda(b)$  as irreducible constituent of  $S(a)R'(b)$ ;

$$(1.4) \quad S(a)R'(b) \leftrightarrow \sum_{\kappa, \lambda} c_{\kappa\lambda} (F_\kappa(a) \times F'_\lambda(b))$$

(the sign  $\leftrightarrow$  indicates that two representations have the same irreducible constituents).

**Lemma 1.** Let  $A \supset A_1 \supset A_2 \supset \dots \supset A_r = 0$  be a composition series of two-sided ideals of  $A$ .

1) If  $A_i e_\lambda \supset A_{i+1} e_\lambda$ , then the  $A$ -left-module  $A_i e_\lambda / A_{i+1} e_\lambda$  is simple and  $A_i e_\mu = A_{i+1} e_\mu$  for  $\mu \neq \lambda$ .

2) If  $e_\kappa A_i \supset e_\kappa A_{i+1}$ , then the  $A$ -right-module  $e_\kappa A_i / e_\kappa A_{i+1}$  is simple and  $e_\nu A_i = e_\nu A_{i+1}$  for  $\nu \neq \kappa$ .

3) If  $A_i e_\lambda / A_{i+1} e_\lambda \cong \bar{A} \bar{e}_\kappa$ , then  $e_\kappa A_i / e_\kappa A_{i+1} \cong \bar{e}_\lambda \bar{A}$ , and conversely.

*Proof.* If  $F_\kappa(a) \times F'_\lambda(b)$  belongs to the  $A$ -two-sided module  $A_i / A_{i+1}$ , then the  $A$ -left-module  $A_i / A_{i+1}$  is a direct sum of  $f(\lambda)$  simple left-moduli isomorphic to  $\bar{A} \bar{e}_\kappa$ , and the  $A$ -right-module  $A_i / A_{i+1}$  is a direct sum of  $f(\kappa)$  simple right-moduli isomorphic to  $\bar{e}_\lambda \bar{A}$ :

$$\begin{aligned} A_i / A_{i+1} &\cong \mathfrak{M}_1 + \mathfrak{M}_2 + \dots + \mathfrak{M}_{f(\lambda)}, & \mathfrak{M}_s &\cong \bar{A} \bar{e}_\kappa \\ A_i / A_{i+1} &\cong \mathfrak{N}_1 + \mathfrak{N}_2 + \dots + \mathfrak{N}_{f(\kappa)}, & \mathfrak{N}_t &\cong \bar{e}_\lambda \bar{A}. \end{aligned}$$

Since  $A_i$  is a direct sum:  $A_i = A_i E_1 + A_i E_2 + \dots + A_i E_n$ , we have for the  $A$ -left-module  $A_i / A_{i+1}$

$$\begin{aligned} A_i / A_{i+1} &= A_i E_1 / A_{i+1} E_1 + A_i E_2 / A_{i+1} E_2 + \dots + A_i E_n / A_{i+1} E_n \\ &= (A_i / A_{i+1}) E_1 + (A_i / A_{i+1}) E_2 + \dots + (A_i / A_{i+1}) E_n. \end{aligned}$$

While we have

$$(A_i / A_{i+1}) E_\mu \cong (\mathfrak{N}_1 + \mathfrak{N}_2 + \dots + \mathfrak{N}_{f(\kappa)}) E_\mu = 0 \quad (\mu \neq \lambda)$$

since  $\mathfrak{N}_t E_\mu \cong \bar{e}_\lambda \bar{A} E_\mu = 0$ . This shows that  $A_i E_\mu = A_{i+1} E_\mu$  for  $\mu \neq \lambda$  and

$$\begin{aligned} A_i / A_{i+1} &= A_i E_\lambda / A_{i+1} E_\lambda \\ &= A_i e_{\lambda, 1} / A_{i+1} e_{\lambda, 1} + \dots + A_i e_{\lambda, f(\lambda)} / A_{i+1} e_{\lambda, f(\lambda)}. \end{aligned}$$

We then have  $A_i e_\lambda / A_{i+1} e_\lambda \cong \bar{A} \bar{e}_\kappa$  and  $A_i e_\mu = A_{i+1} e_\mu$  for  $\mu \neq \lambda$ . Similarly, we have for the  $A$ -right-module  $A_i / A_{i+1}$

$$A_i / A_{i+1} = E_\kappa A_i / E_\kappa A_{i+1}, \quad E_\nu A_i / E_\nu A_{i+1} = 0 \quad (\nu \neq \kappa).$$

Hence  $e_\kappa A_i / e_\kappa A_{i+1} \cong \bar{e}_\lambda \bar{A}$  and  $e_\nu A_i = e_\nu A_{i+1}$  for  $\nu \neq \kappa$ . This completes the proof.

From the composition series of  $A$  in Lemma 1, we have

$$A e_\lambda \cong A_i e_\lambda \cong \dots \cong A_r e_\lambda = 0.$$

We can choose a subsequence  $A e_\lambda = B_0 e_\lambda, B_1 e_\lambda, \dots, B_{m(\lambda)} e_\lambda = 0$  from this sequence such that  $B_i e_\lambda \supset B_{i+1} e_\lambda$  and every  $A_j e_\lambda$  is equal to one of them. According to Lemma 1

$$A e_\lambda \supset B_1 e_\lambda \supset \dots \supset B_{m(\lambda)} e_\lambda = 0$$

is a composition series of the left ideal  $Ae_\lambda$ . We obtain readily from Lemma 1 the following

**Theorem 1.** *Let  $c_{\kappa\lambda}$  denote multiplicity of  $F_\kappa(a) \times F'_\lambda(b)$  in  $S(a)R'(b)$ . Then*

$$\begin{aligned} 1) \quad & \begin{cases} U_\lambda(a) \leftrightarrow \sum_{\kappa} c_{\kappa\lambda} F_\kappa(a) \\ V_\kappa(a) \leftrightarrow \sum_{\lambda} c_{\kappa\lambda} F'_\lambda(a) \end{cases} \\ 2) \quad & S(a)R'(b) \leftrightarrow \sum_{\lambda} U_\lambda(a) \times F'_\lambda(b) \leftrightarrow \sum_{\kappa} F_\kappa(a) \times V'_\kappa(b). \end{aligned}$$

Theorem 1 shows that the  $c_{\kappa\lambda}$  are the *Cartan invariants* of  $A$ . Let  $m(\lambda)$  and  $n(\kappa)$  be the lengths of composition series of  $Ae_\lambda$  and  $e_\lambda A$ . Then

$$\sum_{\lambda} m(\lambda) = \sum_{\kappa} n(\kappa) = \sum_{\kappa, \lambda} c_{\kappa\lambda} = r.$$

**Corollary.** *If  $A$  is semi-simple, then*

$$(1.5) \quad S(a)R'(b) \cong \sum_{\kappa} F_\kappa(a) \times F'_\kappa(b).$$

As one can easily see, we can replace in Lemma 1  $A$  by any two-sided ideal  $\mathfrak{A}$  of  $A$ . Hence, if  $U_\lambda^*$  and  $V_\kappa^*$  are the representations of  $A$  belonging to  $\mathfrak{A}e_\lambda$  and  $e_\kappa \mathfrak{A}$ , then

$$U_\lambda \cong \begin{pmatrix} G_\lambda & \\ * & U_\lambda^* \end{pmatrix}, \quad V_\kappa \cong \begin{pmatrix} H_\kappa & * \\ & V_\kappa^* \end{pmatrix}$$

and

$$(1.6) \quad \begin{cases} U_\lambda^*(a) \leftrightarrow \sum_{\kappa} h_{\kappa\lambda} F_\kappa(a) \\ V_\kappa^*(a) \leftrightarrow \sum_{\lambda} h_{\kappa\lambda} F'_\lambda(a). \end{cases}$$

2. Let  $B$  and  $C$  be two subalgebras of  $A$  having the unit element 1 in common with  $A$ . Define  $E_\kappa^{(1)}$  and  $e_\kappa^{(1)}$  of  $B$  in the same way as we defined  $E_\lambda$  and  $e_\lambda$  of  $A$ . Let  $\bar{B}$  be the residue class algebra of  $B$  with respect to its radical, and let  $\bar{e}_\kappa^{(1)}$  be the residue class containing  $e_\kappa^{(1)}$ . Let  $F_1^{(1)}, F_2^{(1)}, \dots, F_t^{(1)}$  be the distinct irreducible representations of  $B$ . We denote by  $U_1^{(1)}, U_2^{(1)}, \dots, U_t^{(1)}$  and  $V_1^{(1)}, V_2^{(1)}, \dots, V_t^{(1)}$  the indecomposable constituents of the left and the right regular representations of  $B$  respectively, where  $U_\kappa^{(1)}$  and  $V_\kappa^{(1)}$  belong to  $B\bar{e}_\kappa^{(1)}$  and  $\bar{e}_\kappa^{(1)}B$ . We call the representations of  $A$  belonging to  $Ae_\kappa^{(1)}$  and  $e_\kappa^{(1)}A$ , the induced representations of  $A$  from  $U_\kappa^{(1)}$  and  $V_\kappa^{(1)}$ ,

and denote by  $\tilde{U}_\kappa^{(1)}$  and  $\tilde{V}_\kappa^{(1)}$ . Similarly we can define  $F_\lambda^{(2)}$ ,  $U_\lambda^{(2)}$ ,  $V_\lambda^{(2)}$ ,  $\tilde{U}_\lambda^{(2)}$  and  $\tilde{V}_\lambda^{(2)}$  ( $\lambda = 1, 2, \dots, m$ ) with respect to  $C$ . Let  $S(a)$  and  $R(a)$  have the same meaning as in section 1. Then  $b \times c' \rightarrow S(b)R'(c)$  for  $b \in B$  and  $c \in C$ , is a representation of the direct product  $B \times C'$  and belongs to the  $B$ - $C$ -double-module  $A$ . The distinct irreducible representations of  $B \times C'$  are given by  $F_\kappa^{(1)}(b) \times (F_\lambda^{(2)}(c))'$  ( $\kappa = 1, 2, \dots, l$ ;  $\lambda = 1, 2, \dots, m$ ). Corresponding to Lemma 1, we have

**Lemma 2.** *Let  $A \supset M_1 \supset M_2 \supset \dots \supset M_s = 0$  be a composition series of  $B$ - $C$ -double-module  $A$ . Then*

1) *If  $M_i e_\lambda^{(2)} \supset M_{i+1} e_\lambda^{(2)}$ , then the  $B$ -left-module  $M_i e_\lambda^{(2)} / M_{i+1} e_\lambda^{(2)}$  is simple and  $M_i e_\mu^{(2)} = M_{i+1} e_\mu^{(2)}$  for  $\mu \neq \lambda$ .*

2) *If  $e_\kappa^{(1)} M_i \supset e_\kappa^{(1)} M_{i+1}$ , then the  $C$ -right-module  $e_\kappa^{(1)} M_i / e_\kappa^{(1)} M_{i+1}$  is simple and  $e_\nu^{(1)} M_i = e_\nu^{(1)} M_{i+1}$  for  $\nu \neq \kappa$ .*

3) *If  $M_i e_\lambda^{(2)} / M_{i+1} e_\lambda^{(2)} \cong \bar{B} e_\lambda^{(1)}$ , then  $e_\kappa^{(1)} M_i / e_\kappa^{(1)} M_{i+1} \cong e_\lambda^{(2)} \bar{C}$ , and conversely.*

*Proof.* If  $F_\kappa^{(1)}(b) \times (F_\lambda^{(2)}(c))'$  belongs to the  $B$ - $C$ -double-module  $M_i / M_{i+1}$  then the  $B$ -left-module  $M_i / M_{i+1}$  is a direct sum of  $f_2(\lambda)$  simple left-moduli isomorphic to  $\bar{B} e_\lambda^{(1)}$ , and the  $C$ -right-module  $M_i / M_{i+1}$  is a direct sum of  $f_1(\kappa)$  simple right-moduli isomorphic to  $e_\lambda^{(2)} \bar{C}$ :

$$\begin{aligned} M_i / M_{i+1} &\cong \mathfrak{S}_1 + \mathfrak{S}_2 + \dots + \mathfrak{S}_{f_2(\lambda)}, & \mathfrak{S}_\mu &\cong \bar{B} e_\mu^{(1)} \\ M_i / M_{i+1} &\cong \mathfrak{I}_1 + \mathfrak{I}_2 + \dots + \mathfrak{I}_{f_1(\kappa)}, & \mathfrak{I}_\nu &\cong e_\nu^{(2)} \bar{C}. \end{aligned}$$

Since  $M_i$  is a direct sum:

$$\begin{aligned} M_i &= M_i E_1^{(2)} + M_i E_2^{(2)} + \dots + M_i E_m^{(2)} \\ &= E_1^{(1)} M_i + E_2^{(1)} M_i + \dots + E_m^{(1)} M_i, \end{aligned}$$

we have in a quite similar manner as Lemma 1

$$\begin{aligned} M_i / M_{i+1} &= M_i E_\lambda^{(2)} / M_{i+1} E_\lambda^{(2)}, & M_i E_\mu^{(2)} / M_{i+1} E_\mu^{(2)} &= 0 & (\mu \neq \lambda) \\ M_i / M_{i+1} &= E_\kappa^{(1)} M_i / E_\kappa^{(1)} M_{i+1}, & E_\nu^{(1)} M_i / E_\nu^{(1)} M_{i+1} &= 0 & (\nu \neq \kappa). \end{aligned}$$

Hence we obtain readily our assertions.

As an immediate consequence we have

**Theorem 2.** *Let  $\sigma_{\kappa\lambda}$  denote the multiplicity of  $F_\kappa^{(1)}(b) \times (F_\lambda^{(2)}(c))'$  in  $S(b)R'(c)$ . Then*

$$1) \quad \begin{cases} \tilde{U}_\lambda^{(2)}(b) \leftrightarrow \sum_{\kappa} \sigma_{\kappa\lambda} F_\kappa^{(1)}(b) & (\text{for } b \in B) \\ \tilde{V}_\kappa^{(1)}(c) \leftrightarrow \sum_{\lambda} \sigma_{\kappa\lambda} F_\lambda^{(2)}(c) & (\text{for } c \in C) \end{cases}$$

$$2) \quad S(b)R'(c) \leftrightarrow \sum_{\kappa} F_{\kappa}^{(1)}(b) \times (\tilde{V}_{\kappa}^{(1)}(c))' \leftrightarrow \sum_{\lambda} \tilde{U}_{\lambda}^{(2)}(b) \times (F_{\lambda}^{(2)}(c))'.$$

**Corollary.** Let  $\sigma_{\lambda\kappa}^*$  denote the multiplicity of  $F_{\lambda}^{(2)}(c) \times (F_{\kappa}^{(1)}(b))'$  in  $S(c)R'(b)$ . Then

$$1) \quad \begin{cases} \tilde{U}_{\kappa}^{(1)}(c) \leftrightarrow \sum_{\lambda} \sigma_{\lambda\kappa}^* F_{\lambda}^{(2)}(c) & (\text{for } c \in C) \\ \tilde{V}_{\lambda}^{(2)}(b) \leftrightarrow \sum_{\kappa} \sigma_{\lambda\kappa}^* F_{\kappa}^{(2)}(b) & (\text{for } b \in B) \end{cases}$$

$$2) \quad S(c)R'(b) \leftrightarrow \sum_{\lambda} F_{\lambda}^{(2)}(c) \times (\tilde{V}_{\lambda}^{(2)}(b))' \leftrightarrow \sum_{\kappa} \tilde{U}_{\kappa}^{(1)}(c) \times (F_{\kappa}^{(1)}(b))'.$$

In particular, for  $C = A$ , we have the following relations<sup>1)</sup>

$$(2.1) \quad \begin{cases} U_{\lambda}(b) \leftrightarrow \sum_{\kappa} \pi_{\kappa\lambda} F_{\kappa}^{(1)}(b) & (\text{for } b \in B) \\ \tilde{V}_{\kappa}^{(1)}(a) \leftrightarrow \sum_{\lambda} \pi_{\kappa\lambda} F_{\lambda}(a) & (\text{for } a \in A) \end{cases}$$

$$(2.2) \quad \begin{cases} \tilde{U}_{\kappa}^{(1)}(a) \leftrightarrow \sum_{\lambda} \pi_{\lambda\kappa}^* F_{\lambda}(a) & (\text{for } a \in A) \\ V_{\lambda}(b) \leftrightarrow \sum_{\kappa} \pi_{\lambda\kappa}^* F_{\kappa}^{(1)}(b) & (\text{for } b \in B) \end{cases}$$

$$(2.3) \quad \begin{cases} S(b)R'(a) \leftrightarrow \sum_{\kappa} F_{\kappa}^{(2)}(b) \times (\tilde{V}_{\kappa}^{(1)}(a))' \leftrightarrow \sum_{\lambda} U_{\lambda}(b) \times F_{\lambda}'(a) \\ S(a)R'(b) \leftrightarrow \sum_{\lambda} F_{\lambda}(a) \times V_{\lambda}'(b) \leftrightarrow \sum_{\kappa} \tilde{U}_{\kappa}^{(1)}(a) \times (F_{\kappa}^{(1)}(b))'. \end{cases}$$

Further, for  $C = B$ , we have

$$(2.4) \quad \begin{cases} \tilde{U}_{\lambda}^{(1)}(b) \leftrightarrow \sum_{\kappa} \omega_{\kappa\lambda} F_{\kappa}^{(1)}(b) \\ \tilde{V}_{\kappa}^{(1)}(b) \leftrightarrow \sum_{\lambda} \omega_{\kappa\lambda} F_{\lambda}^{(1)}(b) \end{cases} \quad (\text{for } b \in B).$$

Theorem 1 and (2.3) yield

$$(2.5) \quad S(a)R'(b) \leftrightarrow \sum_{\lambda} U_{\lambda}(a) \times F_{\lambda}'(b) \leftrightarrow \sum_{\kappa} \tilde{U}_{\kappa}^{(1)}(a) \times (F_{\kappa}^{(1)}(b))'.$$

If we set  $B^* = B/(B \cap N)$ , then  $B^*$  is a subalgebra of  $\bar{A} = A/N$ . Since  $B \cap N$  is contained in the radical of  $B$ ,  $B^*$  has the same irreducible representations  $F_{\kappa}^{(1)}$  ( $\kappa = 1, 2, \dots, l$ ) with  $B$ . Let  $U_{\kappa}^*$  be the indecomposable constituent of the left regular representation of  $B^*$  corresponding to  $F_{\kappa}^{(1)}$ . Then we get

$$U_{\kappa}^{(1)} \cong \begin{pmatrix} U_{\kappa}^* & \\ * & W_{\kappa} \end{pmatrix}.$$

1) Cf. Nakayama (11) p. 335.



Further, let us denote by  $\tilde{U}_\kappa^*$  the representation of  $\bar{A}$  induced from  $U_\kappa^*$ . If  $\tilde{U}_\kappa^{(\omega)}(a) \cong \sum_\lambda \alpha_{\kappa\lambda} U_\lambda(a)$  for  $a \in A$ , then

$$(2.6) \quad \tilde{U}_\kappa^*(a) \cong \sum_\lambda \alpha_{\kappa\lambda} F_\lambda(a).$$

(2.5), applied to  $\bar{A}$  and its subalgebra  $B^*$ , gives

$$\bar{S}(a)\bar{R}'(b) \leftrightarrow \sum_\lambda F_\lambda(a) \times F'_\lambda(b) \leftrightarrow \sum_\kappa \tilde{U}_\kappa^*(a) \times (F_\kappa^{(\omega)}(b))'$$

where  $\bar{S}(\bar{a})$  and  $\bar{R}(\bar{a})$  (for  $\bar{a} \in \bar{A}$ ) are the regular representations of  $\bar{A}$ . We then have from (2.6)

$$F_\lambda(b) \leftrightarrow \sum_\kappa \alpha_{\kappa\lambda} F_\kappa^{(\omega)}(b).$$

Hence we have formulas

$$(2.7) \quad \begin{cases} F_\lambda(b) \leftrightarrow \sum_\kappa \alpha_{\kappa\lambda} F_\kappa^{(\omega)}(b) & (\text{for } b \in B) \\ \tilde{U}_\kappa^{(\omega)}(a) \leftrightarrow \sum_\lambda \alpha_{\kappa\lambda} U_\lambda(a) & (\text{for } a \in A). \end{cases}$$

Similarly we obtain  $\tilde{V}_\kappa^{(\omega)}(a) \cong \sum_\lambda \alpha_{\kappa\lambda} V_\lambda(a)$  with the same  $\alpha_{\kappa\lambda}$ .

## II. Ordinary representations of groups.

3. Let  $\Gamma(\mathfrak{G})$  be the group ring of a group  $\mathfrak{G}$  over an algebraically closed field  $K$  of characteristic 0:

$$\Gamma(\mathfrak{G}) = G_1K + G_2K + \dots + G_gK, \quad G_1 = 1$$

where  $G_1, G_2, \dots, G_g$  are the elements of  $\mathfrak{G}$ . Instead of considering representations of  $\mathfrak{G}$ , we may consider representations of  $\Gamma(\mathfrak{G})$ . Let  $Z_1, Z_2, \dots, Z_u$  be the distinct irreducible representations of  $\mathfrak{G}$ . To each  $Z_i$  there corresponds a contragredient (irreducible) representation  $G \rightarrow Z_i(G^{-1})$  ( $G \in \mathfrak{G}$ ) which we denote by  $Z_{i'}$ . Let  $S(G)$  and  $R(G)$  be the left and the right regular representations of  $\Gamma(\mathfrak{G})$  defined by a basis  $G_1, G_2, \dots, G_g$ . Then

$$G_s \times G_t \rightarrow S(G_s)R'(G_t^{-1})$$

is a representation of the direct product  $\mathfrak{G} \times \mathfrak{G}$ . Since  $\Gamma(\mathfrak{G})$  is semi-simple, we have from (1.5)

$$(3.1) \quad S(G_s)R'(G_t^{-1}) \cong \sum_i Z_i(G_s) \times Z_{i'}(G_t).$$

Let  $C_1, C_2, \dots, C_m$  be the classes of conjugate elements in  $\mathfrak{G}$  and let  $n_\nu$  be the order of the normalizer  $\mathfrak{N}(G)$  of an element  $G$  contained in  $C_\nu$ . Then  $g_\nu = g/n_\nu$  denotes the number of elements in  $C_\nu$ . Denote by  $C_\nu^*$  the class containing the elements reciprocal to those of  $C_\nu$ .

**Theorem 3.** *Let  $\psi(G_s \times G_t)$  be the character of the representation  $S(G_s)R'(G_t^{-1})$  of  $\mathfrak{G} \times \mathfrak{G}$ . Then*

$$\psi(G_s \times G_t) = n_\nu \delta_{\nu\mu} \quad (\text{for } G_s \in G_\nu, G_t \in C_\mu).$$

*Proof.* From  $G_s(G_1 G_2 \dots G_\theta)G_t^{-1} = (G_1 G_2 \dots G_\theta)S(G_s)R'(G_t^{-1})$ , we have  $S(G_s)R'(G_t^{-1}) = (\alpha_{kt}(G_s \times G_t))$  where  $(\alpha_{kt}(G_s \times G_t))$  possesses one 1 in each column and row. If  $G_k^{-1}G_s G_k \neq G_t$  for any  $G_k$ , then  $G_s G_k G_t^{-1} \neq G_k$ , hence  $\alpha_{kk}(G_s \times G_t) = 0$  for any  $k$ . This implies that  $\psi(G_s \times G_t) = 0$  for  $\nu \neq \mu$ . Now we consider the case when  $G_t = G_s$ .  $G_s G_k G^{-1} = G_k$ , that is,  $\alpha_{kk}(G_s \times G_s) = 1$  if and only if  $G_k$  lies in  $\mathfrak{N}(G_s)$ . Hence we have  $\psi(G_s \times G_s) = n_\nu$ . Finally suppose that  $G_s$  and  $G_t$  are conjugate in  $\mathfrak{G}$ . From  $G_t = G_r^{-1}G_s G_r$ , we find

$$\begin{aligned} S(G_s)R'(G_t^{-1}) &= S(G_s)R'(G_r)R'(G_s^{-1})R'(G_r^{-1}) \\ &= R'(G_r)S(G_s)R'(G_s^{-1})(R'(G_r))^{-1}. \end{aligned}$$

This shows that  $\psi(G_s \times G_t) = \psi(G_s \times G_s) = n_\nu$ .

We denote by  $\chi_i$  the character of  $Z_i$ . The value of a character  $\chi_i$  for the class  $C_\nu$  will be indicated by  $\chi_i^{(\nu)}$ . From (3.1) and Theorem 3, we have the orthogonality relation for ordinary group characters:

$$(3.2) \quad \sum_i \chi_i^{(\nu)} \chi_i^{(\mu)*} = n_\nu \delta_{\nu\mu}.$$

We arrange  $\chi_i^{(\nu)}$  in matrix form  $Z = (\chi_i^{(\nu)})$  ( $i$  row index,  $\nu$  column index). Then (3.2) becomes

$$(3.3) \quad Z'Z = (n_\nu \delta_{\nu\mu}) = T.$$

Since  $T$  in (3.3) is non-singular, we obtain  $u = m$  by a well known manner. The number of distinct (absolutely) irreducible representations is equal to the number of classes of conjugate elements in  $\mathfrak{G}$ . We can derive from (3.2)

$$(3.4) \quad \sum_\nu g_\nu \chi_i^{(\nu)} \chi_j^{(\nu)*} = g \delta_{ij} \quad (i, j = 1, 2, \dots, u).$$

Further, (3.4) yields

$$(3.5) \quad \sum_{G \in \mathfrak{G}} \chi_i(G) = \begin{cases} g & (i = 1) \\ 0 & (i \neq 1). \end{cases}$$

Here,  $\chi_1$  means the character of the 1-representation.

4. Let  $\mathfrak{H}$  and  $\mathfrak{J}$  be two subgroups of  $\mathfrak{G}$ , and denote by  $\xi_1, \xi_2, \dots, \xi_s$  the irreducible characters of  $\mathfrak{H}$  and by  $\zeta_1, \zeta_2, \dots, \zeta_t$  those of  $\mathfrak{J}$ . Let  $\tilde{\xi}_\lambda$  and  $\tilde{\zeta}_i$  be the characters of  $\mathfrak{G}$  induced from  $\xi_\lambda$  and  $\zeta_i$ . From Theorem 2 we have

$$(4.1) \quad \begin{cases} \tilde{\zeta}_i(H) = \sum_{\lambda} k_{i\lambda} \xi_\lambda(H) & (\text{for } H \in \mathfrak{H}) \\ \tilde{\xi}_\lambda(J) = \sum_i k_{i\lambda} \zeta_i(J) & (\text{for } J \in \mathfrak{J}). \end{cases}$$

In particular, for  $\mathfrak{J} = \mathfrak{G}$ , we have following Frobenius' theorem on induced characters:

$$(4.2) \quad \begin{cases} \chi_i(H) = \sum_{\lambda} l_{i\lambda} \xi_\lambda(H) & (\text{for } H \in \mathfrak{H}) \\ \tilde{\xi}_\lambda(G) = \sum_i l_{i\lambda} \chi_i(G) & (\text{for } G \in \mathfrak{G}). \end{cases}$$

Further, from (2.4) we have

$$(4.3) \quad \begin{cases} \tilde{\xi}_\lambda(H) = \sum_{\kappa} q_{\kappa\lambda} \xi_\kappa(H) \\ \tilde{\xi}_\kappa(H) = \sum_{\lambda} q_{\kappa\lambda} \xi_\lambda(H) \end{cases} \quad (\text{for } H \in \mathfrak{H}).$$

From (4.2) it follows that

$$\tilde{\xi}_\lambda(H) = \sum_i l_{i\lambda} \chi_i(H) = \sum_i \left( \sum_{\kappa} l_{i\kappa} l_{\kappa\lambda} \right) \xi_\kappa(H).$$

Then (4.3) yields  $q_{\kappa\lambda} = \sum_i l_{i\kappa} l_{i\lambda}$ , or in matrix form

$$(4.4) \quad Q = L'L$$

where  $Q = (q_{\kappa\lambda})$ ,  $L = (l_{i\kappa})$ . Theorem 3 and (2.3) yield for  $H \in \mathfrak{H}$

$$(4.5) \quad \sum \tilde{\xi}_\kappa(G) \xi_\kappa(H^{-1}) = \begin{cases} n(H) & \text{for } C(G) = C(H) \\ 0 & \text{for } C(G) \neq C(H) \end{cases}$$

where  $C(G)$  denotes the class of conjugate elements in  $\mathfrak{G}$  which contains  $G$ , and where  $n(G)$  denotes the order of the normalizer  $\mathfrak{N}(G)$ . From (4.3) we obtain

$$\begin{aligned}\sum_{H \in \mathfrak{H}} \tilde{\xi}_\kappa(H) \xi_\lambda(H^{-1}) &= \sum_{H \in \mathfrak{H}} \sum_{\rho} q_{\kappa\rho} \xi_\rho(H) \xi_\lambda(H^{-1}) \\ &= \sum_{\rho} q_{\kappa\rho} \sum_{H \in \mathfrak{H}} \xi_\rho(H) \xi_\lambda(H^{-1}) = q_{\kappa\lambda} h\end{aligned}$$

where  $h$  denotes the order of  $\mathfrak{H}$ . Hence

$$\sum_{\kappa} \sum_{H \in \mathfrak{H}} \tilde{\xi}_\kappa(H) \xi_\kappa(H^{-1}) = \sum_{\kappa} q_{\kappa\kappa} h = \sum_{H \in \mathfrak{H}} n(H).$$

Consequently we have

$$(4.6) \quad \text{tr}(q_{\kappa\lambda}) = \sum_{H \in \mathfrak{H}} n(H) / h.$$

Let us denote by  $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_k$  the classes of conjugate elements in  $\mathfrak{G}$  which contain an element of  $\mathfrak{H}$ , and let  $H_1, H_2, \dots, H_k$  ( $H_i \in \mathfrak{H}$ ) be a complete system of representatives for these classes.

**Theorem 4.** *The number of linearly independent characters of  $\mathfrak{G}$  induced from the  $s$  distinct irreducible characters  $\xi_\kappa$  of  $\mathfrak{H}$  is equal to the number  $k$  of those  $\bar{C}_i$  which contain an element of  $\mathfrak{H}$ .*

*Proof.* If we arrange  $\tilde{\xi}_\kappa(H_m)$  and  $\xi_\kappa(H_m^{-1})$  in matrix form

$$W = (\tilde{\xi}_\kappa(H_m)), \quad U = (\xi_\kappa(H_m^{-1}))$$

( $\kappa$  row index;  $m$  column index). Then (4.5) becomes

$$W'U = (n(H_m)\delta_{mn}) = S.$$

Since  $S$  is non-singular, the rank of  $W$  is equal to  $k$ . But we have  $\tilde{\xi}_\kappa(G) = 0$  for every  $G \notin \bar{C}_i$  ( $i = 1, 2, \dots, k$ ), whence the number of linearly independent characters among  $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_s$  is equal to  $k$ .

5. Let  $\bar{\mathfrak{G}}$  be a group isomorphic to  $\mathfrak{G}$  by correspondence  $G_m \rightarrow \bar{G}_m$ . Then the elements  $G_m \times \bar{G}_m$  ( $m = 1, 2, \dots, g$ ) of the direct product  $\mathfrak{G} \times \bar{\mathfrak{G}}$  form the subgroup  $\mathfrak{G}_0$  isomorphic to  $\mathfrak{G}$ . We can choose  $G_1, G_2, \dots, G_g$  as a complete residue system of  $\mathfrak{G} \times \bar{\mathfrak{G}}$  (mod  $\mathfrak{G}_0$ ):

$$\mathfrak{G} \times \bar{\mathfrak{G}} = G_1\mathfrak{G}_0 + G_2\mathfrak{G}_0 + \dots + G_g\mathfrak{G}_0, \quad G_1 = 1.$$

**Lemma 3.** *Let us denote by  $\tilde{D}$  the representation of  $\mathfrak{G} \times \bar{\mathfrak{G}}$  induced from a representation  $D$  of  $\mathfrak{G}_0$ . Then*

$$\tilde{D}(G_m \times \bar{G}_n) \cong S(G_m)R'(G_n^{-1}) \times D(G_n)$$

where  $S(G)$  and  $R(G)$  are the regular representations of  $\mathfrak{G}$ .

*Proof.* We have

$$\tilde{D}(G_m \times \bar{G}_n) \cong (D(G_k(G_m \times \bar{G}_n)G_i^{-1}))_{ki} = (D(G_k G_m G_i^{-1} \times \bar{G}_n))_{ki}$$

where  $D(G_s \times G_t)$  is defined to be the zero matrix for  $G_s \times G_t$  not contained in  $\mathfrak{G}_0$ . If we set  $M(G_m \times \bar{G}_n) = (D(G_k G_m G_i^{-1} \times \bar{G}_n))_{ki}$ , then

$$M(G_m) = (D(G_k G_m G_i^{-1}))_{ki} = S(G_m) \times I_f$$

where  $f$  is the degree of  $D$  and  $I_f$  is the unit matrix of degree  $f$ . Further we have

$$\begin{aligned} M(\bar{G}_n) &= (D(G_k G_i^{-1} \times \bar{G}_n))_{ki} = R'(G_n^{-1}) \times D(G_n \times \bar{G}_n) \\ &= R'(G_n^{-1}) \times D(G_n) \end{aligned}$$

since we get  $G_k G_n^{-1} = G_i$  from  $G_k G_i = G_n$ . Hence

$$\begin{aligned} \tilde{D}(G_m \times \bar{G}_n) &\cong (S(G_m) \times I_f)(R'(G_n^{-1}) \times D(G_n)) \\ &= S(G_m)R'(G_n^{-1}) \times D(G_n). \end{aligned}$$

If we take  $\bar{G}_1, \bar{G}_2, \dots, \bar{G}_q$  as a complete residue system of  $\mathfrak{G} \times \mathfrak{G} \pmod{\mathfrak{G}_0}$ , then we have in a same way,  $S(G_n)R'(G_m^{-1}) \times D(G_m)$  as the representation of  $\mathfrak{G} \times \mathfrak{G}$  induced from  $D$ . Of course we find

$$(5.1) \quad S(G_m)R'(G_n^{-1}) \times D(G_n) \cong S(G_n)R'(G_m^{-1}) \times D(G_m).$$

In particular, for  $G_n = 1$ , we obtain

$$(5.2) \quad S(G) \times I_f \cong R'(G^{-1}) \times D(G) \quad (G \in \mathfrak{G}).$$

We have finally  $S(G) \times I_f \cong S(G) \times D(G)^{\mathfrak{G}}$ , since  $S(G) \cong R'(G^{-1})$ . Further we can see that  $S(G_m)R'(G_n^{-1})$  is the representation of  $\mathfrak{G} \times \mathfrak{G}$ , induced from the 1-representation of  $\mathfrak{G}_0$ .

Let us denote the irreducible characters of  $\mathfrak{G}$  by  $\chi_1, \chi_2, \dots, \chi_u$ . Then the distinct irreducible characters of  $\mathfrak{G} \times \mathfrak{G}$  are given by  $\chi_i(G_m)\chi_j(G_n)$  ( $i, j = 1, 2, \dots, u$ ). Since  $\chi_i(G)\chi_j(G)$  ( $G \in \mathfrak{G}$ ) is a character of  $\mathfrak{G}$ , irreducible or reducible, we obtain formulas

$$(5.3) \quad \chi_i(G)\chi_j(G) = \sum_k a_{ijk}\chi_k(G)$$

where the  $a_{ijk}$  are rational integers,  $a_{ijk} \geq 0$ , and  $a_{ijk} = a_{jik}$ . Let us

1) Cf. Osima (15).

denote by  $\tilde{\chi}_k$  the character of  $\mathfrak{G} \times \bar{\mathfrak{G}}$  induced from  $\chi_k$  of  $\mathfrak{G}_0$ . Then from Lemma 3

$$(5.4) \quad \tilde{\chi}_k(G_m \times \bar{G}_n) = \sum_i \chi_i(G_m) \chi_{i'}(G_n) \chi_k(G_n)$$

where  $\chi_{i'}$  is the character contragredient to  $\chi_i$ .

**Theorem 5.** *If  $\chi_i(G) \chi_j(G) = \sum_k a_{ijk} \chi_k(G)$ , then  $\chi_{i'}(G) \chi_k(G) = \sum_j a_{ijk} \chi_j(G)$ , that is,  $a_{ijk} = a_{i'kj}$ .*

*Proof.* (4.2) applied to  $\mathfrak{G} \times \bar{\mathfrak{G}}$  and its subgroup  $\mathfrak{G}_0$ , gives

$$\tilde{\chi}_k(G_m \times \bar{G}_n) = \sum_{i,j} a_{ijk} \chi_i(G_m) \chi_j(G_n).$$

Hence, by (5.4) we have

$$\sum_i \chi_i(G_m) \chi_{i'}(G_n) \chi_k(G_n) = \sum_{i,j} a_{ijk} \chi_i(G_m) \chi_j(G_n).$$

Since  $\chi_1, \chi_2, \dots, \chi_u$  are linearly independent, it follows that

$$\chi_{i'}(G_n) \chi_k(G_n) = \sum_j a_{ijk} \chi_j(G_n).$$

**Theorem 6.** *Let  $\chi_i^{(\nu)}$  be the value of  $\chi_i$  for the class  $C_\nu$  of conjugate elements in  $\mathfrak{G}$ . Then*

$$\sum_\nu \chi_i^{(\nu)} \chi_j^{(\nu*)} = \sum_{k,l} a_{ikl} a_{jkl}.$$

*Proof.* From (5.3) and Theorem 5, it follows that

$$\begin{aligned} \sum_k \chi_k^{(\nu)} \chi_k^{(\nu)} \chi_i^{(\nu)} &= \sum_k \sum_l a_{ikl} \chi_l^{(\nu)} \chi_k^{(\nu)} = \sum_{k,l} \left( \sum_m a_{ikl} a_{k'lm} \chi_m^{(\nu)} \right) \\ &= \sum_{k,l} \left( \sum_m a_{ikl} a_{mkl} \chi_m^{(\nu)} \right). \end{aligned}$$

On the other hand, from (3.2)

$$\sum_k \chi_k^{(\nu)} \chi_k^{(\nu)} \chi_i^{(\nu)} = n_\nu \chi_i^{(\nu)} = g \chi_i^{(\nu)} / g_\nu.$$

Hence

$$\sum_m \left( \sum_{k,l} a_{ikl} a_{mkl} \right) g_\nu \chi_m^{(\nu)} = g \chi_i^{(\nu)}.$$

Here, we multiply by  $\chi_j^{(\nu*)}$ , and add over  $\nu$ , and use (3.4)

$$\sum_{k,l} a_{ikl} a_{jkl} = \sum_\nu \chi_i^{(\nu)} \chi_j^{(\nu*)}.$$

We shall derive some further relations for the  $a_{ijk}$ . By Theorem 5

$$\sum_{k,l} a_{ikl} a_{jkl} = \sum_{k',l} a_{ik'l} a_{jk'l} = \sum_{k',l} a_{k'l} a_{klj}.$$

Thus we have

$$(5.5) \quad \sum_{k,l} a_{ikl} a_{jkl} = \sum_{k,l} a_{kll} a_{klj}.$$

We can also show the following relations

$$\sum_{k,l} a_{ikl} a_{j'kl} = \sum_{k,l} a_{ikl} a_{jlk} = \sum_v \chi_i^{(v)} \chi_j^{(v)}.$$

In particular, from Theorem 6 we find

$$(5.6) \quad \sum_{k,l} a_{ikl}^a = \sum_{k,l} a_{kll}^a = \sum_v \chi_i^{(v)} \chi_i^{(v*)}.$$

$$(5.7) \quad \sum_k \dot{a}_{kk'l} = \sum_k a_{ikl} = \sum_v \chi_i^{(v)}.$$

6. Let  $Z_1, Z_2, \dots, Z_u$  have the same meaning as in section 3. Denote by  $X_1, X_2, \dots, X_s$  the distinct irreducible representations of a subgroup  $\mathfrak{H}$  of  $\mathfrak{G}$ .  $H \rightarrow Z_i(H)$  ( $H \in \mathfrak{H}$ ) is a representation of  $\mathfrak{H}$  which we denote by  $Z_i(\mathfrak{H})$ . Now we can distribute  $Z_1, Z_2, \dots, Z_u$  into a certain number of blocks with respect to  $\mathfrak{H}$  by the following manner. We say that  $Z_i$  and  $Z_j$  belong to the same block, if in the sequence

$$Z_i(\mathfrak{H}), Z_k(\mathfrak{H}), \dots, Z_l(\mathfrak{H}), Z_j(\mathfrak{H})$$

any two consecutive  $Z_m(\mathfrak{H})$  have an irreducible constituent in common. Thus  $Z_1, Z_2, \dots, Z_u$  appear distributed into  $r$  " $\mathfrak{H}$ -block"  $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_r$ . Further we say that all the irreducible constituents of  $Z_i(\mathfrak{H})$  belong to  $\mathfrak{B}_\kappa$ , if  $Z_i$  belongs to  $\mathfrak{B}_\kappa$ . Denote by  $\tilde{X}_\lambda$  the representation of  $\mathfrak{G}$  induced from  $X_\lambda$ . Then, as we can easily see, all the irreducible constituents  $Z_i$  of  $\tilde{X}_\lambda$  belong to the same block. Let us set

$$(6.1) \quad \mathfrak{M} = \bigcap_{G \in \mathfrak{G}} G^{-1} \mathfrak{H} G.$$

Then  $\mathfrak{M}(\subset \mathfrak{H})$  is an invariant subgroup of  $\mathfrak{G}$ . Let  $\bar{C}_1, \bar{C}_2, \dots, \bar{C}_l$  be the classes of conjugate elements in  $\mathfrak{G}$  which contain an element of  $\mathfrak{M}$ . We denote by  $\bar{C}_v^*$  the sum of all elements in  $\bar{C}_v$ . Since  $\bar{C}_v^*$  is a sum of complete classes of  $\mathfrak{H}$ , we have

$$(6.2) \quad X_\lambda(G^{-1} \bar{C}_v^* G) = X_\lambda(\bar{C}_v^*) = \rho_\lambda^{(v)} I_{h(\lambda)}$$

where  $h(\lambda)$  is the degree of  $X_\lambda$  and  $I_{h(\lambda)}$  is the unit matrix of degree  $h(\lambda)$ . Let  $A(\mathfrak{G})$  and  $A(\mathfrak{H})$  be the centers of group rings  $\Gamma(\mathfrak{G})$  and  $\Gamma(\mathfrak{H})$ , and let  $\omega_i$  be the character of  $A(\mathfrak{G})$  determined by  $\chi_i$ . Then  $Z_i(\bar{C}_j^*) = \omega_i(\bar{C}_j^*)I_{f_i}$ . Since

$$Z_i(\mathfrak{H}) \cong \begin{pmatrix} X_{\kappa} & & \\ & \ddots & \\ & & X_{\lambda} \end{pmatrix}$$

we have from (6.2)

$$Z_i(\bar{C}_j^*) = \begin{pmatrix} X_{\kappa}(\bar{C}_j^*) & & \\ & \ddots & \\ & & X_{\lambda}(\bar{C}_j^*) \end{pmatrix} = \begin{pmatrix} \rho_{\kappa}^{(\nu)} I_{h(\kappa)} & & \\ & \ddots & \\ & & \rho_{\lambda}^{(\nu)} I_{h(\lambda)} \end{pmatrix}$$

Then it follows that

$$(6.3) \quad \omega_i(\bar{C}_j^*) = \rho_{\kappa}^{(\nu)} = \dots = \rho_{\lambda}^{(\nu)}$$

**Theorem 7.** *The two irreducible representations  $Z_i$  and  $Z_j$  belong to the same  $\mathfrak{H}$ -block if and only if  $\chi_i(M)/f_i = \chi_j(M)/f_j$  for all  $M \in \mathfrak{M}$ .*

*Proof.* Assume that  $Z_i$  and  $Z_j$  belong to the same block. From (6.3) it follows that  $\omega_i(\bar{C}_j^*) = \omega_j(\bar{C}_j^*)$ , whence  $\chi_i(M)/f_i = \chi_j(M)/f_j$ . Now we prove the converse. Let us denote by  $\mathfrak{A}_\lambda$  the set of those elements of  $\Gamma(\mathfrak{G})$  which are represented by 0 in every  $Z_i$  outside of  $\mathfrak{B}_\lambda$ . Then  $\mathfrak{A}_\lambda (\lambda = 1, 2, \dots, r)$  are ideals of  $\Gamma(\mathfrak{G})$ , and  $\Gamma(\mathfrak{G})$  splits into a direct sum:

$$\Gamma(\mathfrak{G}) = \mathfrak{A}_1 + \mathfrak{A}_2 + \dots + \mathfrak{A}_r.$$

We then have

$$\Gamma(\mathfrak{H}) = \Gamma(\mathfrak{H}) \cap \mathfrak{A}_1 + \Gamma(\mathfrak{H}) \cap \mathfrak{A}_2 + \dots + \Gamma(\mathfrak{H}) \cap \mathfrak{A}_r.$$

Let  $\epsilon_\lambda$  be the unit element of  $\Gamma(\mathfrak{H}) \cap \mathfrak{A}_\lambda$ . Then  $\Gamma(\mathfrak{H}) \cap \mathfrak{A}_\lambda = \Gamma(\mathfrak{H}) \epsilon_\lambda$  and  $\Gamma(\mathfrak{G}) \epsilon_\lambda \equiv \mathfrak{A}_\lambda$ . But we find  $\Gamma(\mathfrak{G}) \epsilon_\lambda = \mathfrak{A}_\lambda$ , since  $\Gamma(\mathfrak{G}) = \sum \Gamma(\mathfrak{G}) \epsilon_\lambda$ . This implies that  $\epsilon_\lambda$  is the unit element of  $\mathfrak{A}_\lambda$ , and belongs to  $A(\mathfrak{G})$ . Then  $\epsilon_\lambda$  belongs to  $\Gamma(G^{-1}\mathfrak{H}G)$  for any  $G \in \mathfrak{G}$ , whence  $\epsilon_\lambda$  is in  $\Gamma(\mathfrak{M})$ . Consequently  $\epsilon_\lambda$  belongs to

$$\Gamma(\mathfrak{M}) \cap A(\mathfrak{G}) = K\bar{C}_1^* + K\bar{C}_2^* + \dots + K\bar{C}_t^*$$

and, hence, is expressed by  $a_1\bar{C}_1^* + a_2\bar{C}_2^* + \dots + a_t\bar{C}_t^*$  ( $a_i \in K$ ). Since



$\epsilon_\lambda$  is represented by 1 in all  $Z_i$  of  $\mathfrak{B}_\lambda$ , and is represented by 0 in every  $Z_m$  outside of  $\mathfrak{B}_\lambda$ , there exists at least a class  $\bar{C}_\mu$  such that  $\omega_i(\bar{C}_\mu^*) \neq \omega_m(\bar{C}_\mu^*)$ , i.e.  $\chi_i(M)/f_i \neq \chi_m(M)/f_m$  for  $M \in \bar{C}_\mu$ . This completes the proof.

**Theorem 8.** *The number of  $\mathfrak{S}$ -blocks of  $\mathfrak{G}$  is equal to the number of classes of conjugate elements in  $\mathfrak{G}$  which contain an element of  $\mathfrak{M}$ .*

*Proof.*  $\mathfrak{B}_\lambda (\lambda = 1, 2, \dots, r)$  are in 1-1 correspondence with the irreducible representations of

$$A(\mathfrak{G}) \cap A(\mathfrak{M}) = K\bar{C}_1^* + K\bar{C}_2^* + \dots + K\bar{C}_l^*.$$

Hence we have  $r = l$ .

Let  $\theta_i$  and  $\theta_j$  be two irreducible characters of  $\mathfrak{M}$ , then  $\theta_i$  and  $\theta_j$  are called associated in  $\mathfrak{G}$ , if there exists a fixed element  $G$  such that  $\theta_i(M) = \theta_j(G^{-1}MG)$  ( $M \in \mathfrak{M}$ ). We can distribute the irreducible characters of  $\mathfrak{M}$  into the associated classes. From Theorem 4, the number of the associated classes is equal to  $l$ . Since the irreducible constituents of  $Z_i(\mathfrak{M})$  are associated in  $\mathfrak{G}$ ,  $\mathfrak{S}$ -blocks  $\mathfrak{B}_\lambda (\lambda = 1, 2, \dots, l)$  are in 1-1 correspondence with the associated classes of  $\mathfrak{M}$ .

Let us denote by  $Z_1, Z_2, \dots, Z_l$  a complete system of representatives for  $\mathfrak{S}$ -blocks  $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_l$  and let  $Z_{\lambda,1} = Z_\lambda, Z_{\lambda,2}, \dots, Z_{\lambda,s(\lambda)}$  be the irreducible representations in  $\mathfrak{B}_\lambda$ . We have from (3.4) for  $M_i \in \bar{C}_\nu, M_j \in \bar{C}_\mu$ ,

$$\begin{aligned} \sum_{m=1}^u g(M_i) \chi_m(M_i) \chi_m(M_j) &= \sum_{m=1}^u f_m \omega_m(\bar{C}_\nu^*) \chi_m(M_j) \\ &= \sum_{\kappa=1}^l \omega_\kappa(\bar{C}_\nu^*) \sigma_\kappa(M_j) = g \delta_{\nu\mu}^*. \end{aligned}$$

where  $\sigma_\kappa(M_j) = \sum_{\rho=1}^{s(\kappa)} f_{\kappa\rho} \chi_{\kappa,\rho}(M_j)$  and  $f_{\kappa\rho}$  is the degree of  $Z_{\kappa,\rho}$ . From Theorem 7

$$\sigma_\kappa(M_j) = \left( \sum_{\rho=1}^{s(\kappa)} f_{\kappa\rho}^2 / f_\kappa \right) \chi_\kappa(M_j) = a_\kappa \chi_\kappa(M_j).$$

Hence we have

$$(6.4) \quad \sum_{\kappa} b_\kappa \chi_\kappa(M_i) \chi_\kappa(M_j) = n(M_i) \delta_{\nu\mu}^*$$

where  $b_\kappa = a_\kappa / f_\kappa = \sum_{\rho} f_{\kappa\rho}^2 / f_\kappa^2$ . Further (6.4) yields

$$(6.5) \quad \sum_{M \in \mathfrak{M}} b_\kappa \chi_\kappa(M) \chi_\lambda(M^{-1}) = g \delta_{\kappa\lambda} \quad (\text{for } \chi_\kappa \text{ in } \mathfrak{B}_\kappa, \chi_\lambda \text{ in } \mathfrak{B}_\lambda).$$

### III. Modular representations of groups.

7. We consider representations of  $\mathfrak{G}$  in an algebraically closed field  $\bar{K}$  of characteristic  $p$ . Let  $F_1, F_2, \dots, F_m$  be the distinct irreducible representations and let  $U_1, U_2, \dots, U_m$  be corresponding indecomposable constituents of the (left, for example) regular representation of  $\mathfrak{G}$ . Let us denote by  $\varphi_\lambda$  and  $\eta_\lambda$  the characters of  $F_\lambda$  and  $U_\lambda$ . We understand these characters in the sense of Brauer and Nesbitt<sup>1)</sup>: they are complex numbers and are defined only for the  $p$ -regular elements<sup>2)</sup>. We denote by  $C_1, C_2, \dots, C_t$  the classes of conjugate elements which contain the  $p$ -regular elements. The value of characters  $\varphi_\lambda$  and  $\eta_\lambda$  for the class  $C_\nu$  will be indicated by  $\varphi_\lambda^{(\nu)}$  and  $\eta_\lambda^{(\nu)}$ . Theorem 3, combined with (1.4), yields

$$\sum_{\kappa, \lambda} c_{\kappa\lambda} \varphi_\kappa^{(\nu)} \varphi_\lambda^{(\mu)} = n_\nu \delta_{\nu\mu}^*.$$

Since  $\eta_\lambda^{(\nu)} = \sum_{\kappa} c_{\kappa\lambda} \varphi_\kappa^{(\nu)}$  by Theorem 1, we have

$$(7.1) \quad \sum_{\lambda} \eta_\lambda^{(\nu)} \varphi_\lambda^{(\mu)} = n_\nu \delta_{\nu\mu}^*.$$

We arrange  $\varphi_\lambda^{(\nu)}$  and  $\eta_\lambda^{(\nu)}$  in matrix form  $\phi = (\varphi_\lambda^{(\nu)})$ ,  $H = (\eta_\lambda^{(\nu)})$  ( $\lambda$  row index,  $\nu$  column index). Then (7.1) becomes

$$(7.2) \quad H' \phi = (n_\nu \delta_{\nu\mu}^*) = P.$$

Since  $P$  is non-singular, we get  $m = t$  in a same way as in Brauer and Nesbitt (6), and consequently

$$|H| \neq 0, \quad |\phi| \neq 0.$$

(7.1) yields the following

$$(7.3) \quad \sum_{\nu} g_\nu \varphi_\kappa^{(\nu)} \eta_\lambda^{(\nu*)} = g \delta_{\kappa\lambda} \quad (\kappa, \lambda = 1, 2, \dots, m).$$

As is well known, the ordinary irreducible representation  $Z_i$  determines a modular representation (reducible or irreducible)  $\bar{Z}_i^{(3)}$ . Let  $d_{i\lambda}$  denote the multiplicity of  $F_\lambda$  in  $\bar{Z}_i$ . Brauer and Nesbitt called these  $d_{i\lambda}$  the decomposition numbers of  $\mathfrak{G}$ .

**Theorem 9.** If  $\chi_i = \sum_{\lambda} d_{i\lambda} \varphi_\lambda$ , then  $\eta_\lambda = \sum_i d_{i\lambda} \chi_i$ .

1) See Brauer-Nesbitt (6).

2) By a  $p$ -regular element of  $\mathfrak{G}$ , we understand an element whose order is prime to  $p$ .

3) See Brauer-Nesbitt (6) p. 558.

*Proof.* From (3.2) and (7.1), we have

$$\sum_i \chi_i^{(\nu)} \chi_i^{(\mu)} = \sum_\lambda \eta_\lambda^{(\nu)} \varphi_\lambda^{(\mu)} \quad (\nu, \mu = 1, 2, \dots, m).$$

Hence

$$\begin{aligned} \sum_i \chi_i^{(\nu)} \chi_i^{(\mu)} &= \sum_i \chi_i^{(\nu)} \sum_\lambda d_{i\lambda} \varphi_\lambda^{(\mu)} \\ &= \sum_\lambda \left( \sum_i d_{i\lambda} \chi_i^{(\nu)} \right) \varphi_\lambda^{(\mu)} = \sum_\lambda \eta_\lambda^{(\nu)} \varphi_\lambda^{(\mu)}. \end{aligned}$$

This implies that  $\eta_\lambda = \sum_i d_{i\lambda} \chi_i$ .

**Theorem 10<sup>v</sup>.** If  $\chi_i = \sum_\kappa d_{i\kappa} \varphi_\kappa$  and  $\eta_\lambda = \sum_\kappa c_{\kappa\lambda} \varphi_\kappa$ , then

$$c_{\kappa\lambda} = c_{\lambda\kappa} = \sum_i d_{i\kappa} d_{i\lambda}.$$

*Proof.* According to Theorem 9, we have

$$\eta_\lambda = \sum_i d_{i\lambda} \chi_i = \sum_i d_{i\lambda} \sum_\kappa d_{i\kappa} \varphi_\kappa = \sum_\kappa \left( \sum_i d_{i\kappa} d_{i\lambda} \right) \varphi_\kappa$$

and hence  $c_{\kappa\lambda} = \sum_i d_{i\kappa} d_{i\lambda} = c_{\lambda\kappa}$ .

If we set  $(c_{\kappa\lambda}) = C$ ,  $(d_{i\kappa}) = D$ , then

$$(7.4) \quad C = D' D$$

where  $D'$  is the transpose of  $D$ . From Theorem 1, combined with  $c_{\kappa\lambda} = c_{\lambda\kappa}$ , we can see that  $U_\lambda \leftrightarrow V_\lambda$ , but in virtue of the fact that group ring is symmetric<sup>2)</sup>, we have certainly  $U_\lambda \cong V_\lambda$ .

8. Let  $\mathfrak{H}$  and  $\mathfrak{J}$  be two subgroups of  $\mathfrak{G}$ . Let us denote by  $\varphi_1^*, \varphi_2^*, \dots, \varphi_k^*$  the irreducible characters of  $\mathfrak{H}$  and let  $\eta_1^*, \eta_2^*, \dots, \eta_k^*$  be the corresponding indecomposable characters of the regular representation of  $\mathfrak{H}$ . Similarly we define  $\varphi'_\lambda$  and  $\eta'_\lambda$  ( $\lambda = 1, 2, \dots, l$ ) for  $\mathfrak{J}$ . Further we denote by  $\tilde{\varphi}_\rho^*, \tilde{\eta}_\rho^*, \tilde{\varphi}'_\lambda$  and  $\tilde{\eta}'_\lambda$  the characters of  $\mathfrak{G}$  induced from  $\varphi_\rho^*, \eta_\rho^*, \varphi'_\lambda$  and  $\eta'_\lambda$  respectively. Theorem 2, applied to the group ring of  $\mathfrak{G}$ , yields

$$(8.1) \quad \begin{cases} \tilde{\eta}'_\lambda(H) = \sum_\rho \sigma_{\rho\lambda} \varphi_\rho^*(H) & (\text{for } p\text{-regular elements } H \in \mathfrak{H}) \\ \tilde{\eta}_\rho^*(J) = \sum_\lambda \sigma_{\rho\lambda} \varphi'_\lambda(J) & (\text{for } p\text{-regular elements } J \in \mathfrak{J}). \end{cases}$$

In particular, for  $\mathfrak{J} = \mathfrak{G}$ , from (2.1) and (2.7) we have formulas<sup>3)</sup>

1) H. Nagao has obtained independently a simple proof for this theorem using the properties of the induced characters of  $\mathfrak{G}$ .

2) See Brauer-Nesbitt (5). Cf. also Nakayama-Nesbitt (12).

3) Nakayama (11) p. 366. Brauer-Nesbitt (6) p. 582.

$$(8.2) \quad \begin{cases} \eta_\lambda(H') = \sum_{\rho} \tau_{\rho\lambda} \varphi_\rho^*(H') \\ \widetilde{\eta}_\rho^*(G') = \sum_{\lambda} \tau_{\rho\lambda} \varphi_\lambda(G') \end{cases}$$

$$(8.3) \quad \begin{cases} \varphi_\lambda(H') = \sum_{\rho} \alpha_{\rho\lambda} \varphi_\rho^*(H') \\ \widetilde{\eta}_\rho^*(G') = \sum_{\lambda} \alpha_{\rho\lambda} \eta_\lambda(G') \end{cases}$$

where  $G'$  and  $H'$  mean the  $p$ -regular elements of  $\mathfrak{G}$  and  $\mathfrak{H}$  respectively. Finally, from

$$\sum_{\sigma} \widetilde{\eta}_\sigma^*(G') \varphi_\sigma^*(H') = \sum_{\rho} \widetilde{\varphi}_\rho^*(G') \eta_\rho^*(H')$$

or from (8.3) directly, we have formulas

$$(8.4) \quad \begin{cases} \eta_\lambda(H') = \sum_{\rho} \beta_{\rho\lambda} \eta_\rho^*(H') \\ \widetilde{\varphi}_\rho^*(G') = \sum_{\lambda} \beta_{\rho\lambda} \varphi_\lambda(G'). \end{cases}$$

Let  $\chi_i$  and  $\xi_v$  be the ordinary irreducible characters of  $\mathfrak{G}$  and  $\mathfrak{H}$ . We can prove easily the following formulas

$$(8.5) \quad \begin{cases} \chi_i(H') = \sum_{\rho} m_{i\rho} \varphi_\rho^*(H') \\ \widetilde{\eta}_\rho^*(G') = \sum_i m_{i\rho} \chi_i(G') \end{cases}$$

$$(8.6) \quad \begin{cases} \widetilde{\xi}_v(H') = \sum_{\rho} n_{v\rho} \varphi_\rho^*(H') \\ \widetilde{\eta}_\rho^*(H') = \sum_v n_{v\rho} \xi_v(H') \end{cases}$$

$$(8.7) \quad \begin{cases} \widetilde{\varphi}_\sigma^*(H') = \sum_{\rho} r_{\rho\sigma} \varphi_\rho^*(H') \\ \widetilde{\eta}_\rho^*(H') = \sum_{\sigma} r_{\rho\sigma} \eta_{\sigma'}^*(H'). \end{cases}$$

Further, from (2.4) we have

$$(8.8) \quad \begin{cases} \widetilde{\eta}_\sigma^*(H') = \sum_{\rho} \omega_{\rho\sigma} \varphi_\rho^*(H') \\ \eta_\rho^*(H') = \sum_{\sigma} \omega_{\rho\sigma} \varphi_\sigma^*(H'). \end{cases}$$

If we put

$$W = (\omega_{\rho\sigma}), \quad M = (m_{i\rho}), \quad R = (r_{\rho\sigma}), \quad A = (\alpha_{\rho\lambda}), \quad B = (\beta_{\rho\lambda}),$$

then we obtain from the above formulas

$$(8.9) \quad W = M'M = ACA' = C^*BA' = C^*R'$$

where  $C^* = (c_{\rho\sigma}^*)$  has the same significance for  $\mathfrak{S}$  as  $C$  has for  $\mathfrak{G}$ .

**Theorem 11<sup>1)</sup>.** *The number of linearly independent characters of  $\mathfrak{G}$  induced from the  $k$  distinct irreducible characters  $\varphi_p^*$  of  $\mathfrak{S}$  is equal to the number of the classes of conjugate elements which contain a  $p$ -regular element of  $\mathfrak{S}$ .*

*Proof.* We have for  $p$ -regular elements  $H_i, H_j$  of  $\mathfrak{S}$

$$\sum_p \widehat{\varphi}_p^*(H_i) \eta_p^*(H_j^{-1}) = \begin{cases} n(H_i) & \text{for } C(H_j) = C(H_i) \\ 0 & \text{for } C(H_j) \neq C(H_i) \end{cases}$$

where  $C(H)$  denotes a class of conjugate elements in  $\mathfrak{G}$  which contains  $H$ . Then we can obtain our assertion in the similar way as Theorem 4.

Similarly in section 6, we can distribute the indecomposable representations  $U_1, U_2, \dots, U_m$  into a certain number of blocks with respect to  $\mathfrak{S}$ . We say that  $U_\kappa$  and  $U_\lambda$  belong to the same block, if in the sequence

$$U_\kappa(\mathfrak{S}), U_\mu(\mathfrak{S}), \dots, U_\nu(\mathfrak{S}), U_\lambda(\mathfrak{S})$$

any two consecutive  $U_\rho(\mathfrak{S})$  have an irreducible constituent in common. Thus  $U_1, U_2, \dots, U_m$  appear distributed in  $s$  " $\mathfrak{S}^*$ -blocks"  $\mathfrak{B}_1^*, \mathfrak{B}_2^*, \dots, \mathfrak{B}_s^*$ . We also say that  $F_\kappa$  belongs to  $\mathfrak{B}_\sigma^*$  when  $U_\kappa$  belongs to  $\mathfrak{B}_\sigma^*$ . Then we can see that all the irreducible constituents  $F_\rho$  of  $U_\kappa$  belong to  $\mathfrak{B}_\sigma^*$ . Moreover all the irreducible constituents of the modular representation  $\bar{Z}_i$  of  $\mathfrak{G}$  which is determined by the ordinary irreducible representation  $Z_i$  belong to the same block. If  $\bar{Z}_i$  contains  $F_\kappa$  in  $\mathfrak{B}_\sigma^*$  as its irreducible constituent, then we say that  $Z_i$  also belongs to  $\mathfrak{B}_\sigma^*$ . Let  $\mathfrak{M}$  have the same meaning as in section 6.

**Theorem 12<sup>2)</sup>.** *The ordinary irreducible representations  $Z_i$  and  $Z_j$  belong to the same  $\mathfrak{S}^*$ -block, if and only if*

$$g(M)\chi_i(M)/f_i \equiv g(M)\chi_j(M)/f_j \pmod{\mathfrak{p}}$$

for all  $M \in \mathfrak{M}$ , where  $\mathfrak{p}$  is a fixed prime ideal divisor of  $p$  in  $K^{*3)}$ .

1) Nakayama (11) p. 366.

2) Cf. Brauer-Nesbitt (6) p. 562.

3) We choose the algebraic number field  $K^*$  so that the absolutely irreducible representations of  $\mathfrak{G}$  can be written with coefficients in  $K^*$ .

By Theorem 7, we have

**Corollary.** *If  $Z_i$  and  $Z_j$  belong to the same  $\mathfrak{S}$ -block, then they belong to the same  $\mathfrak{S}^*$ -block.*

9. Let  $\mathfrak{U}$  and  $\mathfrak{U}_0$  have the same meaning as in section 5. Since  $\mathfrak{U}_0$  is isomorphic to  $\mathfrak{U}$ , the characters  $\varphi_\lambda$  and  $\eta_\lambda$  of  $\mathfrak{U}$  may be considered as the characters of  $\mathfrak{U}_0$ . Denote by  $\tilde{\varphi}_\lambda$  and  $\tilde{\eta}_\lambda$  the characters of  $\mathfrak{U} \times \mathfrak{U}$  induced from  $\varphi_\lambda$  and  $\eta_\lambda$  of  $\mathfrak{U}_0$ . Lemma 3 holds also in the modular case, and hence for  $p$ -regular elements  $G_i, G_j$  of  $\mathfrak{U}$ , we have

$$(9.1) \quad \begin{aligned} \tilde{\varphi}_\mu(G_i \times \bar{G}_j) &= \sum_{\kappa} \varphi_\kappa(G_i) \eta_{\kappa'}(G_j) \varphi_\mu(G_j) \\ \tilde{\eta}_\mu(G_i \times \bar{G}_j) &= \sum_{\kappa} \varphi_\kappa(G_i) \eta_{\kappa'}(G_j) \eta_\mu(G_j) \\ &= \sum_{\kappa} \eta_\kappa(G_i) \varphi_{\kappa'}(G_j) \eta_\mu(G_j) \end{aligned}$$

where  $\varphi_{\kappa'}$  and  $\eta_{\kappa'}$  are the characters contragredient to  $\varphi_\kappa$  and  $\eta_\kappa$ .

**Theorem 13<sup>1)</sup>.** *If  $\eta_\kappa(G) \eta_\lambda(G) = \sum_{\mu} \pi_{\kappa\lambda\mu} \varphi_\mu(G)$  for  $p$ -regular elements  $G$  of  $\mathfrak{U}$ , then  $\eta_{\kappa'}(G) \eta_\mu(G) = \sum_{\lambda} \pi_{\kappa\lambda\mu} \varphi_\lambda(G)$ , that is,  $\pi_{\kappa\lambda\mu} = \pi_{\kappa'\mu\lambda}$ .*

*Proof.* Applying (8.2) to  $\mathfrak{U} \times \mathfrak{U}$  and its subgroup  $\mathfrak{U}_0$ , we have

$$\tilde{\eta}_\mu(G_i \times \bar{G}_j) = \sum_{\kappa} \varphi_\kappa(G_i) \eta_{\kappa'}(G_j) \eta_\mu(G_j) = \sum_{\mu, \lambda} \pi_{\kappa\lambda\mu} \varphi_\kappa(G_i) \varphi_\lambda(G_j).$$

This implies that  $\eta_{\kappa'}(G_j) \eta_\mu(G_j) = \sum_{\lambda} \pi_{\kappa\lambda\mu} \varphi_\lambda(G_j)$ .

Further we obtain the following

**Theorem 14.** *For  $p$ -regular elements  $G$  of  $\mathfrak{U}$*

$$(1) \quad \begin{cases} \varphi_\kappa(G) \varphi_\lambda(G) = \sum_{\mu} \alpha_{\kappa\lambda\mu} \varphi_\mu(G) \\ \varphi_{\kappa'}(G) \eta_\mu(G) = \sum_{\lambda} \alpha_{\kappa\lambda\mu} \eta_\lambda(G) \end{cases}$$

$$(2) \quad \begin{cases} \eta_\kappa(G) \eta_\lambda(G) = \sum_{\mu} \beta_{\kappa\lambda\mu} \eta_\mu(G) \\ \eta_{\kappa'}(G) \varphi_\mu(G) = \sum_{\lambda} \beta_{\kappa\lambda\mu} \varphi_\lambda(G). \end{cases}$$

*Proof.* Every indecomposable constituent of the regular representation of  $\mathfrak{U} \times \mathfrak{U}$  is given by  $U_\kappa(G_i) \times U_\lambda(\bar{G}_j)$ . Hence (8.3) and (9.1) yield

1) H. Nagao has proved independently Theorems 13 and 14 by the same manner.

$$\begin{aligned}\widetilde{\eta}_\mu(G_i \times \bar{G}_j) &= \sum_{\kappa} \eta_\kappa(G_i) \varphi_{\kappa'}(G_j) \eta_\mu(G_j) \\ &= \sum_{\kappa, \lambda} \alpha_{\kappa\lambda\mu} \eta_\kappa(G_i) \eta_\lambda(G_j).\end{aligned}$$

This implies that  $\varphi_{\kappa'}(G_j) \eta_\mu(G_j) = \sum_{\lambda} \alpha_{\kappa\lambda\mu} \eta_\lambda(G_j)$ . Similarly from (8.4) we can obtain 2).

In particular, we can see that  $\beta_{\kappa'\lambda 1} = c_{\kappa\lambda}$ . Since (5.2) shows that  $U_\kappa \times D$  splits completely into  $U_1, U_2, \dots, U_m$ , by Theorem 14 we find

$$(9.2) \quad F_{\kappa'}(G) \times U_\mu(G) \cong \sum_{\lambda} \alpha_{\kappa\lambda\mu} U_\lambda(G)^{\nu}.$$

Corresponding to Theorem 6, we have the following formulas

$$(9.3) \quad \begin{aligned}\sum_{\kappa, \mu} \alpha_{\lambda\kappa\mu} \beta_{\rho\kappa\mu} &= \sum_{\kappa, \mu} \beta_{\kappa\mu\lambda} \alpha_{\kappa\mu\rho} = \sum_{\nu} \varphi_\lambda^{(\nu)} \eta_\rho^{(\nu*)} \\ \sum_{\kappa, \mu} \alpha_{\lambda\mu\kappa} \alpha_{\rho\kappa\mu} &= \sum_{\nu} \varphi_\lambda^{(\nu)} \varphi_\rho^{(\nu)} \\ \sum_{\kappa, \mu} \alpha_{\kappa\mu\lambda} \alpha_{\kappa\mu\rho} &= \sum_{\kappa, \mu} \alpha_{\lambda\kappa\mu} \alpha_{\rho\kappa\mu} = \sum_{\nu} \eta_\lambda^{(\nu)} \eta_\rho^{(\nu*)} \\ \sum_{\kappa, \mu} \beta_{\lambda\mu\kappa} \beta_{\rho\kappa\mu} &= \sum_{\nu} \eta_\lambda^{(\nu)} \eta_\rho^{(\nu)}.\end{aligned}$$

From (7.1) we have

$$\begin{aligned}\sum_{\kappa} g_\nu \varphi_\kappa^{(\nu)} \eta_{\kappa'}^{(\nu)} \varphi_\lambda^{(\nu)} &= \sum_{\kappa} \sum_{\mu} \alpha_{\lambda\kappa\mu} g_\nu \varphi_\mu^{(\nu)} \eta_{\kappa'}^{(\nu)} \\ &= \sum_{\sigma} \left( \sum_{\kappa, \mu} \alpha_{\lambda\kappa\mu} \beta_{\sigma\kappa\mu} \right) g_\nu \varphi_\sigma^{(\nu)} = g \varphi_\lambda^{(\nu)}.\end{aligned}$$

Here, we multiply by  $\eta_\rho^{(\nu*)}$ , add over  $\nu$ , and use (7.3)

$$\sum_{\kappa, \mu} \alpha_{\lambda\kappa\mu} \beta_{\rho\kappa\mu} = \sum_{\nu} \varphi_\lambda^{(\nu)} \eta_\rho^{(\nu*)}.$$

On the other hand, we have

$$\sum_{\kappa} \varphi_\kappa^{(\nu)} \eta_{\kappa'}^{(\nu)} \varphi_\lambda^{(\nu)} = \sum_{\kappa} \sum_{\mu} \beta_{\kappa\mu\lambda} \varphi_\mu^{(\nu)} \varphi_\kappa^{(\nu)} = \sum_{\sigma} \left( \sum_{\kappa, \mu} \beta_{\kappa\mu\lambda} \alpha_{\kappa\mu\sigma} \right) \varphi_\sigma^{(\nu)}.$$

Consequently  $\sum_{\kappa, \mu} \alpha_{\lambda\kappa\mu} \beta_{\rho\kappa\mu} = \sum_{\kappa, \mu} \beta_{\kappa\mu\lambda} \alpha_{\kappa\mu\rho}$ . This completes the proof of the first formula (9.3). Now we shall prove the second formula (9.3).

$$\begin{aligned}\sum_{\kappa} g_\nu \varphi_\kappa^{(\nu)} \eta_{\kappa'}^{(\nu)} \varphi_\lambda^{(\nu)} &= \sum_{\kappa} \left( \sum_{\mu'} \alpha_{\lambda'\mu'\kappa'} \eta_{\mu'}^{(\nu)} \right) g_\nu \varphi_\kappa^{(\nu)} \\ &= \sum_{\kappa, \mu'} \alpha_{\lambda'\mu'\kappa'} \sum_{\sigma'} \alpha_{\kappa'\sigma'\mu'} g_\nu \eta_{\sigma'}^{(\nu)} \\ &= \sum_{\sigma} \left( \sum_{\kappa, \mu} \alpha_{\lambda\mu\kappa} \alpha_{\kappa\sigma\mu} \right) g_\nu \eta_{\sigma'}^{(\nu)} = g \varphi_\lambda^{(\nu)}.\end{aligned}$$

1) Cf. Osima (15).

Here, we multiply by  $\varphi_\rho^{(\nu)}$ , add over  $\nu$ , and use (7.3)

$$\sum_{\kappa, \mu} \alpha_{\lambda\mu\kappa} \alpha_{\rho\kappa\mu} = \sum_{\nu} \varphi_\lambda^{(\nu)} \varphi_\rho^{(\nu)}.$$

Similarly  $\sum_{\kappa} \varphi_\kappa^{(\nu)} \eta_{\kappa'}^{(\nu)} \eta_{\lambda'}^{(\nu)} = n_\nu \eta_{\lambda'}^{(\nu)}$  and  $\sum_{\kappa} \varphi_\kappa^{(\nu)} \eta_{\kappa'}^{(\nu)} \eta_{\lambda'}^{(\nu)} = n \eta_{\lambda'}^{(\nu)}$  yield last two formulas (9.3) respectively. Furthermore from (9.3) we find

$$\sum_{\kappa, \mu} \alpha_{\kappa\mu\lambda} \alpha_{\kappa\mu\rho'} = \sum_{\kappa, \mu} \alpha_{\lambda\kappa\mu} \alpha_{\rho'\kappa\mu} = \sum_{\kappa, \mu} \beta_{\lambda\mu\kappa} \beta_{\rho\kappa\mu}.$$

In particular, for  $\lambda = 1$  in (9.3), we have<sup>1)</sup>

$$\begin{aligned} \sum_{\kappa} \beta_{\rho\kappa\kappa} &= \sum_{\kappa, \mu} \beta_{\kappa\mu 1} \alpha_{\kappa\mu\rho} = \sum_{\kappa, \mu} c_{\kappa'\mu} \alpha_{\kappa\mu\rho} \\ &= \sum_{\kappa, \mu} c_{\kappa\mu} \alpha_{\kappa'\mu\rho} = \sum_{\nu} \eta_\rho^{(\nu)} \\ \sum_{\kappa} \alpha_{\rho\kappa\kappa} &= \sum_{\nu} \varphi_\rho^{(\nu)}. \end{aligned}$$

#### IV. Representations of groups by collineations<sup>2)</sup>.

10. Let  $\mathfrak{G}$  be a group of finite order  $g$ . We consider the algebra

$$(10.1) \quad (\mathfrak{r}, \mathfrak{G}) = U_E K + U_P K + \dots + U_Q K, \quad U_E = 1$$

$\mathfrak{G} = \{E, P, \dots, Q\}$  over an algebraically closed field  $K$  in which the multiplication is defined by  $U_P U_Q = r_{P,Q} U_{PQ}$ . Here, the  $r_{P,Q}$  are non-zero elements from  $K$  such that  $r_{P,Q} r_{PQ,R} = r_{P,QR} r_{Q,R}$ .  $(\mathfrak{r}, \mathfrak{G})$  is called the collinear group ring of  $\mathfrak{G}$  with factor set  $\mathfrak{r} = \{r_{P,Q}\}$ . If  $U_P \rightarrow M(P)$  is a representation of  $(\mathfrak{r}, \mathfrak{G})$  by linear transformations, then  $M(P)M(Q) = r_{P,Q} M(PQ)$ . Hence  $P \rightarrow M(P)$  is a representation of  $\mathfrak{G}$  by collineations. In the sequence  $P \rightarrow M(P)$  may be called the representation of  $\mathfrak{G}$  with factor set  $\mathfrak{r}$ . If  $P \rightarrow N(P)$  is a representation of  $\mathfrak{G}$  with factor set  $\mathfrak{r}$ , then  $U_P \rightarrow N(P)$  is certainly a representation of  $(\mathfrak{r}, \mathfrak{G})$ . Hence we may consider representations of  $(\mathfrak{r}, \mathfrak{G})$  instead of considering representations of  $\mathfrak{G}$  with factor set  $\mathfrak{r}$ . The two representations  $P \rightarrow M(P)$  with factor set  $\mathfrak{r}$  and  $P \rightarrow N(P)$  with factor set  $\mathfrak{r}'$ , are called associated if  $M(P) = k_P N(P)$  for all  $P$  where the  $k_P$  are non-zero elements from  $K$ . The factor sets  $\mathfrak{r}$  and  $\mathfrak{r}'$  are also called associated. If  $\mathfrak{r}$  and  $\mathfrak{r}'$  are associated, then we find

1) See Brauer-Nesbitt (6) p. 579.

2) Cf. Schur (17), Tazawa (19).



$$(10.2) \quad r'_{P,Q} = k_P k_Q k_{PQ}^{-1} r_{P,Q}.$$

Associated representations are regarded as not essentially distinct.

Now we assume that the characteristic of  $K$  is 0. Then  $(r, \mathfrak{G})$  is semi-simple. We denote by  $Z_1, Z_2, \dots, Z_k$  the distinct irreducible representations of  $(r, \mathfrak{G})$ . Let  $C_1, C_2, \dots, C_n$  be the classes of conjugate elements in  $\mathfrak{G}$ . A class  $C_v$  is called *regular* with respect to  $r$ , if  $C_v$  contains an element  $P$  such that  $r_{P,Q} = r_{Q,P}$  for any  $Q$  of the normalizer  $\mathfrak{N}(P)$ . We shall denote by  $C_1, C_2, \dots, C_d$  ( $d \leq n$ ) the regular classes. Since  $Z_i$  is an irreducible representation of  $\mathfrak{G}$  with factor set  $r$ , we may set  $\chi_i(U_S) = \chi_i(S)$  where  $\chi_i$  is the character of  $Z_i$ . We then have from  $U_P U_S U_P^{-1} = r_{P,S} r_{PSP^{-1},P}^{-1} U_{PSP^{-1}}$

$$(10.3) \quad \chi_i(S) = r_{P,S} r_{PSP^{-1},P}^{-1} \chi_i(PSP^{-1}).$$

**Lemma 4.** *If  $S$  is contained in a non-regular class, then  $\chi_i(S) = 0$  ( $i = 1, 2, \dots, k$ ).*

*Proof.* From (10.3) we find  $\chi_i(S) = r_{P,S} r_{PSP^{-1},P}^{-1} \chi_i(S)$  for any  $P \in \mathfrak{N}(S)$ . By our assumption there exists  $Q \in \mathfrak{N}(S)$  such that  $r_{Q,S} r_{S,Q}^{-1} \neq 0$ , whence  $\chi_i(S) = 0$ .

Since  $Z'_i(U_P^{-1}) Z'_i(U_Q^{-1}) = r_{P^{-1},Q}^{-1} Z'_i(U_{PQ}^{-1})$ ,  $U_P \rightarrow Z'_i(U_P^{-1})$  is an irreducible representation of  $\mathfrak{G}$  with factor set  $r^{-1} = \{r_{P^{-1},Q}^{-1}\}$ . We call this representation contragredient to  $Z_i$  and denote by  $Z'_i$ . If we denote by  $\chi'_i$  the character of  $Z'_i$ , then from  $U_P^{-1} = r_{P^{-1},P^{-1}}^{-1} U_P^{-1}$  we have  $\chi'_i(P) = \chi_i(U_P^{-1}) = r_{P^{-1},P^{-1}}^{-1} \chi_i(P^{-1})$ .

Let  $S(U_P)$  and  $R(U_P)$  be the left and the right regular representations of  $(r, \mathfrak{G})$  defined by a basis  $U_B, U_P, \dots, U_Q$ . Then  $U_P \times U_S \rightarrow S(U_P) R'(U_S^{-1})$  is a representation of the direct product  $(r, \mathfrak{G}) \times (r^{-1}, \mathfrak{G})$  with factor set  $\{r_{P,Q} r_{S,T}^{-1}\}$ . If we denote by  $\phi(U_P \times U_S)$  the character of  $S(U_P) R'(U_S^{-1})$ , then we have from (1.5)  $\phi(U_P \times U_S) = \sum_i \chi_i(U_P) \chi'_i(U_S^{-1})$ , that is

$$(10.4) \quad \phi(P \times S) = \sum_i \chi_i(P) \chi'_i(S).$$

**Lemma 5.** *If  $P$  is an element of  $\mathfrak{N}(S)$ , then  $P \rightarrow r_{S,P} r_{P,S}^{-1}$  is a linear representation of  $\mathfrak{N}(S)$ .*

*Proof.* We have  $U_S U_P U_S^{-1} = r_{S,P} r_{P,S}^{-1} U_P$ ,  $U_S U_Q U_S^{-1} = r_{S,Q} r_{Q,S}^{-1} U_Q$  ( $P, Q \in \mathfrak{N}(S)$ ), whence  $U_S U_P U_Q U_S^{-1} = r_{S,P} r_{P,S}^{-1} r_{S,Q} r_{Q,S}^{-1} U_P U_Q$ . Then  $U_S U_{PQ} U_S^{-1} = r_{S,P} r_{P,S}^{-1} r_{S,Q} r_{Q,S}^{-1} U_{PQ}$ . On the other hand, since  $PQ \in \mathfrak{N}(S)$ , we find  $U_S U_{PQ} U_S^{-1} = r_{S,PQ} r_{PQ,S}^{-1} U_{PQ}$ . Thus we obtain  $(r_{S,P} r_{P,S}^{-1})(r_{S,Q} r_{Q,S}^{-1}) = r_{S,PQ} r_{PQ,S}^{-1}$ .

**Lemma 6.** *Let  $n(S)$  be the order of the normalizer  $\mathfrak{N}(S)$ . Then*

$$\sum_{P \in \mathfrak{N}(S)} r_{S,P} r_{P,S}^{-1} = \begin{cases} n(S) & \text{for } S \text{ in the regular classes} \\ 0 & \text{for } S \text{ in the non-regular classes.} \end{cases}$$

*Proof.* It follows readily from Lemma 5 and (3.5).

$$\textbf{Lemma 7. } \phi(S \times S) = \sum_{P \in \mathfrak{N}(S)} r_{S,P} r_{P,S}^{-1}.$$

*Proof.* Since  $U_S(U_B U_P \dots U_Q)U_S^{-1} = (U_B U_P \dots U_Q)S(U_S)R'(U_S^{-1})$ , we have our assertion in the similar manner as Theorem 3.

If  $S_i$  and  $S_j$  are not conjugate in  $\mathfrak{G}$ , then as one can easily see  $\phi(S_i \times S_j) = 0$ . Further if  $S$  is contained in a non-regular class, then from Lemmas 6 and 7,  $\phi(S \times S) = 0$ . Let  $S_1, S_2, \dots, S_d$  be a complete system of representatives for the regular classes. Then we have from above consideration

$$(10.5) \quad \phi(S_\nu \times S_\lambda) = n(S_\nu) \delta_{\nu\lambda}.$$

Consequently (10.4) yields

$$(10.6) \quad \sum_i \chi_i(S_\nu) \chi_{i'}(S_\lambda) = n(S_\nu) \delta_{\nu\lambda} \quad (\nu, \lambda = 1, 2, \dots, d).$$

If we set

$$Z = (\chi_i(S_\nu)), \quad Y = (\chi_{i'}(S_\nu))$$

( $\nu$  row index:  $i$  column index). Then (10.5) becomes  $Y'Z = (n(S_\nu) \delta_{\nu\lambda}) = V$ . Since  $V$  is non-singular, we have  $k \geq d$ . Suppose that  $k > d$ . Then  $\chi_1(S_\nu), \chi_2(S_\nu), \dots, \chi_k(S_\nu)$  ( $\nu = 1, 2, \dots, d$ ) are linearly dependent:

$$\sum_i a_i \chi_i(S_\nu) = 0.$$

From (10.3) and Lemma 4 we can see that  $\sum_i a_i \chi_i(P) = 0$  for any  $P \in \mathfrak{G}$ . But such a relation is impossible, whence we have  $k = d$ . The number of the distinct irreducible representations of  $\mathfrak{G}$  with factor set  $\tau$  is equal to the number of the regular classes of conjugate elements in  $\mathfrak{G}$ . From (10.6) we can derive as usual

$$(10.7) \quad \sum_{\nu=1}^d g_\nu \chi_i(S_\nu) \chi_{i'}(S_\nu) = g \delta_{ij}$$

where  $g_\nu = g/n(S_\nu)$ .

Let  $(s, \mathfrak{G})$  be the group ring of  $\mathfrak{G}$  with factor set  $s = \{s_{P,q}\}$ :

$$(s, \mathfrak{G}) = V_B K + V_P K + \dots + V_Q K.$$

We denote by  $S_r(U_P)$ ,  $R_r(U_P)$  and  $S_s(V_P)$ ,  $R_s(V_P)$  the regular representations of  $(r, \mathfrak{G})$  and  $(s, \mathfrak{G})$ . Let  $Z_1^{(r)}, Z_2^{(r)}, \dots, Z_t^{(r)} : Z_1^{(s)}, Z_2^{(s)}, \dots, Z_m^{(s)}$  and  $Z_1^{(t)}, Z_2^{(t)}, \dots, Z_q^{(t)}$  ( $t = rs$ ) be the distinct irreducible representations of  $(r, \mathfrak{G})$ ,  $(s, \mathfrak{G})$ , and  $(t, \mathfrak{G})$ <sup>1)</sup>. The characters of  $Z_i^{(r)}$ ,  $Z_j^{(s)}$  and  $Z_k^{(t)}$  we denote by  $\chi_i^{(r)}$ ,  $\chi_j^{(s)}$  and  $\chi_k^{(t)}$ . Since  $Z_i^{(r)}(U_P) \times Z_j^{(s)}(V_P)$  is a representation of an algebra  $(t, \mathfrak{G})$ , we have

$$(10.8) \quad \chi_i^{(r)}(P)\chi_j^{(s)}(P) = \sum_k b_{ijk}\chi_k^{(t)}(P).$$

$$\mathfrak{A} = (U_E \times V_E)K + (U_P \times V_P)K + \dots + (U_Q \times V_Q)K$$

is a subalgebra of  $(r, \mathfrak{G}) \times (s, \mathfrak{G})$  and is isomorphic to  $(t, \mathfrak{G}) = W_E K + W_P K + \dots + W_Q K$ . Hence we may denote  $\mathfrak{A}$  by  $(t, \mathfrak{G})$ .

**Lemma 8.** *If  $\tilde{D}_t$  is the representation of  $(r, \mathfrak{G}) \times (s, \mathfrak{G})$  induced from a representation  $D_t$  of  $(t, \mathfrak{G})$ , then*

$$\tilde{D}_t(U_P \times V_S) \cong S_r(U_P)R_r^*(U_S^{-1}) \times D_t(W_S).$$

*Proof.* We can prove in the similar manner as Lemma 3.

Since  $\tilde{D}_t$  is representation of  $\mathfrak{G} \times \mathfrak{G}$  with factor set  $\{r_{P,Q} s_{S,T}\}$ , we may write

$$\tilde{D}_t(P \times S) \cong S_r(P)R_r^*(S) \times D_t(S)$$

where  $R_r^*$  is the representation of  $\mathfrak{G}$  contragredient to  $R_r$ . Lemma 8 yields

$$S_r(P)R_r^*(S) \times D_t(S) \cong S_s(S)R_s^*(P) \times D_t(P).$$

In particular, for  $S = E$ , we have  $S_r(P) \times I_T \cong R_s^*(P) \times D_t(P)$ . Applying (4.2) to the direct product  $(r, \mathfrak{G}) \times (s, \mathfrak{G})$  and its subalgebra  $(t, \mathfrak{G})$ , we obtain the following

**Theorem 15.** *Let*

$$\chi_i^{(r)}(P)\chi_j^{(s)}(P) = \sum_k b_{ijk}\chi_k^{(t)}(P).$$

*Then*

$$\chi_i^{(r)}(P)\chi_k^{(t)}(P) = \sum_j b_{ijk}\chi_j^{(s)}(P).$$

1) If  $r = \{r_{P,Q}\}$  and  $s = \{s_{P,Q}\}$  are the two factor sets, then  $\{t_{P,Q}\} = \{r_{P,Q}s_{P,Q}\}$  is also a factor set.

11. We shall study briefly the modular representations of  $\mathfrak{G}$  with factor set<sup>1)</sup>. Let  $(\sigma, \mathfrak{G})$  be the collinear group ring with factor set  $\sigma = \{\sigma_{P,Q}\}$  over an algebraically closed field  $\bar{K}$  of characteristic  $p$ . We set

$$g = g'p^a, \quad (g', p) = 1.$$

Any factor set  $\{\sigma_{P,Q}\}$  is associated with a factor set  $\{\rho_{P,Q}\}$  such that  $\rho_{P,Q}^p = 1$  then necessarily  $\rho_{P',Q}^{q'} = 1$  for any  $P, Q \in \mathfrak{G}$ . In the sequence we may only consider such factor set  $\rho$ . Corresponding to a factor set  $\rho$ , a factor set  $r = \{r_{P,Q}\}$  is defined as complex numbers:

$$\bar{r}_{P,Q} = \rho_{P,Q}, \quad r_{P',Q}^{q'} = 1$$

in the same manner as the modular characters were defined. A class  $C_\nu$  is regular with respect to  $\rho$  if and only if  $C_\nu$  is regular with respect to  $r$ . Let  $C_1^*, C_2^*, \dots, C_s^*$  be the regular classes which contain an element whose order is prime to  $p$ . We denote by  $F_1, F_2, \dots, F_t$  the distinct irreducible representations of  $(\rho, \mathfrak{G})$  and by  $U_1, U_2, \dots, U_t$  the corresponding indecomposable constituents of the regular representation of  $(\rho, \mathfrak{G})$ . If  $\varphi_\lambda$  and  $\eta_\lambda$  are the characters of  $F_\lambda$  and  $U_\lambda$ , then the modular characters of  $F_\lambda$  and  $U_\lambda$  are  $\bar{\varphi}_\lambda$  and  $\bar{\eta}_\lambda$  (residue classes mod  $p$ ). We set  $\varphi_\lambda(U_Q) = \varphi_\lambda(Q)$  and  $\eta_\lambda(U_Q) = \eta_\lambda(Q)$ . Now we have

**Lemma 9.** *Let  $Q$  be an element whose order is prime to  $p$ . If  $Q$  is contained in a non-regular class, then  $\varphi_\lambda(Q) = 0$  (and hence  $\bar{\varphi}_\lambda(Q) = 0$ ).*

*Proof.* We can prove in the similar manner as Lemma 4.

Let  $Q_1, Q_2, \dots, Q_s$  be a complete system of representatives for the classes  $C_\nu^*$  ( $\nu = 1, 2, \dots, s$ ). We have from Theorem 1 and Lemma 7

$$(11.1) \quad \sum_{\lambda} \eta_\lambda(Q_\nu) \varphi_{\lambda'}(Q_\mu) = n(Q_\nu) \delta_{\nu\mu} \quad (\nu, \mu = 1, 2, \dots, s)$$

where  $\varphi_{\lambda'}$  is the character contragredient to  $\varphi_\lambda$ . By (11.1) we have  $t \geq s$ . Now suppose that  $t > s$ . Then the modular characters  $\bar{\varphi}_1(Q_\nu), \bar{\varphi}_2(Q_\nu), \dots, \bar{\varphi}_t(Q_\nu)$  ( $\nu = 1, 2, \dots, s$ ) are linearly dependent:

1) See Asano-Osima-Takahasi (1).

2) The value of these characters are complex numbers as in section 7 and are defined only for  $U_Q$  where  $Q$  has an order prime to  $p$ .

$$\sum_{\lambda} \alpha_{\lambda} \bar{\varphi}_{\lambda}(Q_{\nu}) = 0.$$

Then from Lemma 9 it follows that  $\sum_{\lambda} \alpha_{\lambda} \bar{\varphi}_{\lambda}(Q) = 0$  for any element  $Q$  whose order is prime to  $p$ , whence we have finally  $\sum_{\lambda} \alpha_{\lambda} \bar{\varphi}_{\lambda}(G) = 0$  for any element  $G$  of  $\mathfrak{G}$ . But such a relation is impossible. Hence  $t = s$ . The number of the distinct irreducible representations of  $\mathfrak{G}$  with factor set  $\rho$  is equal to the number of regular classes of conjugate elements in  $\mathfrak{G}$  which contain elements of an order prime to  $p$ . It follows from (11.1) that

$$(11.2) \quad \sum_{\nu} g_{\nu} \gamma_{\kappa}(Q_{\nu}) \varphi_{\lambda'}(Q_{\nu}) = g \delta_{\kappa\lambda}.$$

Corresponding to Theorem 15, we have the following formulas for  $Q \in C_{\nu}^*$

$$(11.3) \quad \begin{cases} \gamma_{\kappa}^{(\rho)}(Q) \gamma_{\lambda}^{(\sigma)}(Q) = \sum_{\mu} \tau_{\kappa\lambda\mu}^* \varphi_{\mu}^{(\tau)}(Q) \\ \gamma_{\kappa'}^{(\rho)}(Q) \gamma_{\mu}^{(\tau)}(Q) = \sum_{\lambda} \tau_{\kappa\lambda\mu}^* \varphi_{\lambda}^{(\sigma)}(Q) \end{cases}$$

$$(11.4) \quad \begin{cases} \varphi_{\kappa}^{(\rho)}(Q) \varphi_{\lambda}^{(\sigma)}(Q) = \sum_{\mu} \alpha_{\kappa\lambda\mu}^* \varphi_{\mu}^{(\tau)}(Q) \\ \varphi_{\kappa'}^{(\rho)}(Q) \gamma_{\mu}^{(\tau)}(Q) = \sum_{\lambda} \alpha_{\kappa\lambda\mu}^* \gamma_{\lambda}^{(\sigma)}(Q) \end{cases}$$

$$(11.5) \quad \begin{cases} \gamma_{\kappa}^{(\rho)}(Q) \gamma_{\lambda}^{(\sigma)}(Q) = \sum_{\mu} \beta_{\kappa\lambda\mu}^* \gamma_{\mu}^{(\tau)}(Q) \\ \gamma_{\kappa'}^{(\rho)}(Q) \varphi_{\mu}^{(\tau)}(Q) = \sum_{\lambda} \beta_{\kappa\lambda\mu}^* \varphi_{\lambda}^{(\sigma)}(Q) \end{cases}$$

where  $\tau_{P,Q} = \rho_P, Q \sigma_{P,Q}$ . In particular, we can prove from (11.4)

$$(11.6) \quad F_{\kappa'}^{(\rho)}(G) \times U_{\mu}^{(\tau)}(G) \cong \sum_{\lambda} \alpha_{\kappa\lambda\mu}^* U_{\lambda}^{(\sigma)}(G).$$

#### BIBLIOGRAPHY

- [1] K. ASANO, M. OSIMA and M. TAKAHASI, Über die Darstellung von Gruppen durch Kollineationen im Körper der Charakteristik  $p$ , Proc. Phys. Math. Soc. Japan, 19 (1937).
- [2] R. BRAUER, Über die Darstellung von Gruppen in Galois'schen Feldern, Act. sci. et indust., 195 (1935).
- [3] ———, On modular and  $p$ -adic representations of algebras, Proc. Nat. Acad. Sci., 25 (1939).

1) See Asano-Osima-Takahasi (1) p. 206.

- [4] R. BRAUER and C. NESBITT, On the modular representations of groups of finite order, Univ. Toronto Studies, Math. Series, 4 (1937).
- [5] ———, On the regular representations of algebras, Proc. Nat. Acad. Sci., 23 (1937).
- [6] ———, On the modular characters of groups, Ann. of Math., 42 (1941).
- [7] W. BURNSIDE, The theory of groups of finite order, Cambridge, 1911.
- [8] L. E. DICKSON, Modular theory of group-matrices, Trans. Amer. Math. Soc., 8 (1907).
- [9] ———, Modular theory of group characters, Bull. Amer. Math. Soc., 13 (1907).
- [10] ———, Modern algebraic theories, Chicago, 1926.
- [11] T. NAKAYAMA, Some studies on regular representations, induced representations and modular representations, Ann. of Math., 39 (1938).
- [12] T. NAKAYAMA and C. NESBITT, Note on symmetric algebras, Ann. of Math., 39 (1938).
- [13] C. NESBITT, On the regular representations of algebras, Ann. of Math., 39 (1938).
- [14] E. NOETHER, Hyperkomplexe Größen und Darstellungstheorie, Math. Zeitschr., 30 (1929).
- [15] M. OSIMA, Note on the Kronecker product of representations of a group, Proc. Imp. Acad. Tokyo, 17 (1941).
- [16] I. SCHUR, Neue Begründung der Theorie der Gruppencharaktere, Sitzungsber. Preuss. Akad., (1905).
- [17] ———, Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen, Crelles Journ., 127 (1904), 132 (1907).
- [18] A. SPEISER, Theorie der Gruppen von endlichen Ordnung, Berlin, 1927.
- [19] M. TAZAYA, Über die Darstellung der endlichen verallgemeinerten Gruppen, Sci. Reports Tohoku Imp. Univ., 23 (1934).
- [20] B. L. VAN DER WAERDEN, Gruppen von linearen Transformationen, Berlin, 1935.

DEPARTMENT OF MATHEMATICS,  
OKAYAMA UNIVERSITY

(Received January 10, 1951)