

Mathematical Journal of Okayama University

Volume 33, Issue 1

1991

Article 5

JANUARY 1991

Bialgebras and Galois extensions

Atsushi Nakajima*

*Okayama University

Copyright ©1991 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

BIALGEBRAS AND GALOIS EXTENSIONS

Dedicated to Professor Takasi Nagahara on his 60th birthday

ATSUSHI NAKAJIMA

We shall work over a commutative ring k . A *bialgebra* $B = (B, \mu, \eta, \Delta, \varepsilon)$ is simultaneously a k -algebra with multiplication μ and unit η , and a k -coalgebra with comultiplication Δ and counit ε which are algebra morphisms. A bialgebra B is called a *left Hopf algebra* if it has a morphism $\lambda: B \rightarrow B$ such that

$$\sum_{(x)} \lambda(x_{(1)}) x_{(2)} = \mu(\lambda \otimes \text{Id}) \Delta(x) = \varepsilon(x) \quad (x \in B),$$

where $\otimes = \otimes_k$, $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$ and Id is the identity morphism. λ is called a *left antipode* ([5], [10]). A right Hopf algebra is similarly defined. A left and right Hopf algebra is a Hopf algebra in the usual sense ([12]).

Let S be a k -algebra. S is called a *right B -comodule algebra* if there exists a k -algebra morphism $\alpha_s: S \rightarrow S \otimes B$ such that $(\alpha_s \otimes \text{Id}) \alpha_s = (\text{Id} \otimes \Delta) \alpha_s$ and $(\text{Id} \otimes \varepsilon) \alpha_s = \text{Id}$. The *k -subalgebra of invariants* is defined by $R = S_0 = \{s \in S \mid \alpha_s(s) = s \otimes 1\}$. According to Y. Doi and M. Takeuchi [4], we call that S/R is a *right B -extension*. A left B -extension is defined by using a left comodule algebra instead of the right. A right B -extension S/R is called a *right B -Galois extension* if the morphism $\gamma_s: S \otimes_R S \rightarrow S \otimes B$ defined by $\gamma_s(s \otimes t) = \sum_{(t)} s t_{(0)} \otimes t_{(1)}$ is bijective, where $\alpha_s(t) = \sum_{(t)} t_{(0)} \otimes t_{(1)}$. Since B is a right B -comodule algebra with the structure morphism Δ and the invariant k -subalgebra is $B_0 = k$, B/k is a right B -extension. Then by [3, Cor. 6], B is a right B -Galois extension if and only if B has the antipode (that is, B is a Hopf algebra).

In this paper we consider a right B -Galois extension for a bialgebra B which is not a Hopf algebra. In section 1, we discuss a relation of right and left antipode of a bialgebra B to a right B -Galois extension. In section 2, we assume that there exists a right B -Galois extension S/R . Under the condition, we give some sufficient conditions that a bialgebra B becomes a Hopf algebra. And in section 3, if B is a semigroup bialgebra which is generated by a finite cyclic semigroup (not a group), then there does not exist

This research is partially supported by Grant-in-Aid for Scientific Research (No. 01540051), Ministry of Education, Science and Culture.

a right B -Galois extension and a right $B^* = \text{Hom}_k(B, k)$ -Galois extension of k .

Throughout the following, we will fix the above notations and terminologies. All algebras, morphisms, $\otimes$, Hom , etc. are taken over k unless otherwise stated and we freely use a sigma notation in the sense of M. Sweedler [12].

1. The antipode and the morphism γ_B . We discuss a relation of a right (resp. a left) antipode to a left (resp. a right) invertibility of γ_B .

Theorem 1.1. *Let B be a bialgebra and let γ_B be the morphism from $B \otimes B$ to $B \otimes B$ defined by $\gamma_B(x \otimes y) = \sum_{(y)} xy_{(1)} \otimes y_{(2)}$. Then*

- (1) *B has a right antipode if and only if γ_B has a left inverse.*
- (2) *If B has a left antipode, then γ_B has a right inverse.*
- (3) *If γ_B is an isomorphism, then B has a left antipode and thus B has the antipode.*

Proof. (1) Let λ be a right antipode. We define a morphism $\beta: B \otimes B \rightarrow B \otimes B$ by $\beta(x \otimes y) = \sum_{(y)} x\lambda(y_{(1)}) \otimes y_{(2)}$ ($x, y \in B$). Then

$$\beta\gamma_B(x \otimes y) = \sum_{(y)} xy_{(1)}\lambda(y_{(2)}) \otimes y_{(3)} = \sum_{(y)} x\varepsilon(y_{(1)}) \otimes y_{(2)} = x \otimes y,$$

and so $\beta\gamma_B = \text{Id}$.

Conversely, let $\beta: B \otimes B \rightarrow B \otimes B$ be a left inverse of γ_B . Since β is a left B -module morphism, we can set $\beta(1 \otimes x) = \sum_i x_{1i} \otimes x_{2i}$. By $1 \otimes x = \beta\gamma_B(1 \otimes x) = \sum_{(x), i} x_{(1)}x_{(2)1i} \otimes x_{(2)2i}$, we have

$$\varepsilon(x) = \sum_{(x), i} x_{(1)}x_{(2)1i} \varepsilon(x_{(2)2i}).$$

Define a morphism $\lambda: B \rightarrow B$ by $\lambda(x) = \sum_i x_{1i} \varepsilon(x_{2i})$. Then it is easy to see that

$$\mu(\text{Id} \otimes \lambda) \Delta(x) = \sum_{(x), i} x_{(1)}x_{(2)1i} \varepsilon(x_{(2)2i}) = \varepsilon(x).$$

Thus λ is a right antipode.

(2) If λ is a left antipode of B , then the morphism $\beta: B \otimes B \rightarrow B \otimes B$ defined by $\beta(x \otimes y) = \sum_{(y)} x\lambda(y_{(1)}) \otimes y_{(2)}$ satisfies the relation $\gamma_B\beta = \text{Id}$.

(3) Assume that β is the inverse morphism of γ_B . For any $f, g \in \text{Hom}(B, B)$, we define morphisms

$$\Phi, \Psi: \text{Hom}(B, B) \rightarrow \text{Hom}(B, B)$$

by

$$\Phi(f)(x) = \sum_i x_{i(1)} f(x_{i(2)}) \quad \text{and} \quad \Psi(g)(x) = \sum_i x_{1i} g(x_{2i}),$$

where $\beta(1 \otimes x) = \sum_i x_{1i} \otimes x_{2i}$. By $\gamma_B \beta = Id$, we have

$$(1.1) \quad 1 \otimes x = \gamma_B \beta(1 \otimes x) = \sum_{\substack{(x_{2i}), i}} x_{1i} x_{2i(1)} \otimes x_{2i(2)}.$$

Then by (1.1),

$$((\Psi\Phi)(f))(x) = \sum_i x_{1i} \Phi(f)(x_{2i}) = \sum_{\substack{(x_{1i}), i}} x_{1i} x_{2i(1)} f(x_{2i(2)}) = f(x)$$

and so $\gamma_B \beta = Id$ implies $\Psi\Phi = Id$. Conversely if $\beta\gamma_B = Id$, then

$$1 \otimes x = \beta\gamma_B(1 \otimes x) = \sum_{\substack{(x_{1i}), i}} x_{(1)} x_{(2)1i} \otimes x_{(2)2i},$$

which shows that

$$(1.2) \quad g(x) = \sum_{\substack{(x_{1i}), i}} x_{(1)} x_{(2)1i} g(x_{(2)2i}).$$

Therefore

$$((\Phi\Psi)(g))(x) = \sum_{\substack{(x)}} x_{(1)} \Psi(g)(x_{(2)}) = \sum_{\substack{(x_{1i}), i}} x_{(1)} x_{(2)1i} g(x_{(2)2i}) = g(x).$$

Now we define $\lambda: B \rightarrow B$ by $\lambda(x) = \sum_i x_{1i} \varepsilon(x_{2i})$, and consider the morphism $\mu(\lambda \otimes Id) \Delta$ in $\text{Hom}(B, B)$. An easy calculation shows

$$\Phi(\mu(\lambda \otimes Id) \Delta)(x) = \sum_{\substack{(x)}} x_{(1)} \mu(\lambda \otimes Id) \Delta(x_{(2)}) = \sum_{\substack{(x_{1i}), i}} x_{(1)} x_{(2)1i} \varepsilon(x_{(2)2i}) x_{(3)}.$$

In (1.2), if we take $g = \varepsilon$, then

$$\varepsilon(x) = \sum_{\substack{(x_{1i}), i}} x_{(1)} x_{(2)1i} \varepsilon(x_{(2)2i}).$$

Thus

$$\begin{aligned} \Phi(\mu(\lambda \otimes Id) \Delta)(x) &= \sum_{\substack{(x_{1i}), i}} x_{(1)} x_{(2)1i} \varepsilon(x_{(2)2i}) x_{(3)} = \sum_{\substack{(x)}} \varepsilon(x_{(1)}) x_{(2)} \\ &= x = \sum_{\substack{(x)}} x_{(1)} \varepsilon(x_{(2)}) = \Phi(\varepsilon)(x). \end{aligned}$$

Since $\Psi\Phi = Id$, we have $\mu(\lambda \otimes Id) \Delta = \varepsilon$. This shows that λ is a left antipode and by (1), λ is itself a right antipode.

By Th. 1.1, B is a right B -Galois extension of k if and only if B has the antipode. This was proved by Y. Doi and M. Takeuchi in [3, Cor. 6].

By the similar proof to Th. 1.1, we have the following :

Theorem 1.2. *Let B be a bialgebra and let ${}_B\gamma$ be the morphism from $B \otimes B$ to $B \otimes B$ defined by ${}_B\gamma(x \otimes y) = \Delta(x)(1 \otimes y) = \sum_{(x)} x_{(1)} \otimes x_{(2)}y$. Then*

- (1) *B has a left antipode if and only if ${}_B\gamma$ has a left inverse.*
- (2) *If B has a right antipode, then ${}_B\gamma$ has a right inverse.*
- (3) *B has the antipode if and only if ${}_B\gamma$ has a two-sided inverse.*

For finitely generated k -modules M and N , if $f: M \rightarrow N$ is an epimorphism, then by [11, Th. 1], f is an isomorphism. Therefore by Th. 1.1(2) and Th. 1.2(2), we have the following result which was proved by J. A. Green, W. D. Nichols and E. J. Taft in [5, Prop. 5].

Corollary 1.3. *Let B be a left (or right) Hopf algebra. If B is a finitely generated k -module, then B is a Hopf algebra.*

2. Quadratic bialgebras and Galois extensions. In [7], H. K. Kreimer determined a Hopf algebra which is a free k -module of rank 2 as follows: Let B be a k -bialgebra which is a free k -module of rank 2. Since $\varepsilon: B \rightarrow k$ is a k -module epimorphism, we have $B = k \oplus \text{Ker}(\varepsilon)$. In general it is known that M is a free k -module of rank 1 if and only if $k \oplus M$ is a free k -module of rank 2 (cf. [7, Lemma 1]). Hence there exists a free basis $\{1, \theta\}$ of B such that $\text{Ker}(\varepsilon) = k\theta$. Since $\varepsilon(\theta^2) = \varepsilon(\theta)^2 = 0$, $\theta^2 = q\theta$ for some $q \in k$. And by $\theta = (\varepsilon \otimes 1)\Delta(\theta) = (1 \otimes \varepsilon)\Delta(\theta)$, $\Delta(\theta) = \theta \otimes 1 + 1 \otimes \theta + p(\theta \otimes \theta)$ for some $p \in k$. Moreover by $\Delta(\theta^2) = \Delta(\theta)^2$ and $\theta^2 = q\theta$, we have

$$(2.1) \quad p^2q^2 + 3pq + 2 = (pq+1)(pq+2) = 0.$$

If B has the antipode λ , then by definition of λ and $\theta^2 = q\theta$, there exists $h \in k$ such that $\lambda(\theta) = h\theta$ and $(-h)(pq+1) = 1$. Therefore

$$(2.2) \quad B \text{ has the antipode if and only if } pq+2 = 0.$$

Now we will discuss the following question. For a bialgebra B (not a Hopf algebra) which is a free k -module of rank 2, does there exist a right B -Galois extension of k ?

Let S be a commutative B -comodule algebra with comodule structure morphism $\alpha: S \rightarrow S \otimes B$. Let $R = S_0$ be the invariant k -subalgebra of S . We assume that S is a free R -module with basis $\{1, x\}$ and set $x^2 = mx + n$

$(m, n \in R)$. Noting that R is invariant under α , α can be considered as an R -module morphism. Then by $(Id \otimes \varepsilon)\alpha = Id$, we can set $\alpha(x) = x \otimes 1 + c(1 \otimes \theta) + d(x \otimes \theta)$ for some $c, d \in R$. Since $(\alpha \otimes Id)\alpha = (Id \otimes \Delta)\alpha$ and α is an R -algebra morphism, we have

$$(2.3) \quad cd = cp$$

$$(2.4) \quad d^2 = dp$$

$$(2.5) \quad cm = c^2q + d^2nq + 2dn$$

$$(2.6) \quad dm = d^2mq + 2(c + cq d + dm)$$

Then, for the ordered basis $\{1 \otimes 1, x \otimes 1, 1 \otimes x, x \otimes x\}$ of $S \otimes_R S$ and $\{1 \otimes 1, x \otimes 1, 1 \otimes \theta, x \otimes \theta\}$ of $S \otimes B$, the matrix representation of the morphism $\gamma_S: S \otimes_R S \rightarrow S \otimes B$ is given by

$$A = \begin{pmatrix} 1 & 0 & 0 & n \\ 0 & 1 & 1 & m \\ 0 & 0 & c & dn \\ 0 & 0 & d & c + dm \end{pmatrix}$$

and the determinant of A is $c(c + dm) - d^2n$. Therefore

γ_S is an isomorphism if and only if $c(c + dm) - d^2n$ is invertible in R .

Now we assume that S/k is a right B -Galois extension. Noting that (2.3), (2.4), (2.5) and the fact that $c(c + dm) - d^2n$ is invertible, we can easily see that

$$\begin{aligned} c(c + dm) - d^2n &= c^2 + cpm - d^2n = c^2 + p(c^2q + d^2nq + 2dn) - d^2n \\ &= c^2(1 + pq) + d^2npq + d^2n \\ &= (c^2 + d^2n)(1 + pq). \end{aligned}$$

Therefore $1 + pq$ is invertible and by (2.1), $pq + 2 = 0$. This means that B has the antipode. These proves the following

Theorem 2.1. *Let B be a bialgebra which is a free k -module of rank 2. Let S be a commutative right B -comodule algebra with invariant k -subalgebra R . If S/R is a right B -Galois extension which is a free R -module of rank 2, then B has the antipode.*

Using the localization, we have

Corollary 2.2. *Let B and S be as in Th. 2.1. If a right B -Galois extension S/R is a finitely generated projective k -module of rank 2, then B has the antipode.*

Next we consider a non-commutative quadratic extension. Let k be a commutative ring generated by 1 and let S be a k -algebra with subalgebra R . Assume that S is a free quadratic extension of R , that is, $S = R \oplus Rx$ and $x^2 = mx + n$ ($m, n \in R$), where $\{1, x\}$ is a free basis. We set

$$xr = \sigma(r)x + D(r) \quad (r \in R).$$

Then it is easy to see that σ is a k -algebra morphism with $\sigma(1) = 1$ and $D(st) = D(s)t + \sigma(s)D(t)$ ($s, t \in R$). If S is a right B -comodule algebra with structure morphism $\alpha: S \rightarrow S \otimes B$ and $S_0 = R$, then α is a left R -linear morphism which is given by

$$\alpha(x) = x \otimes 1 + c \otimes \theta + dx \otimes \theta \quad (c, d \in R),$$

and the relation (2.3) and (2.4) are also satisfied in the same way as to the commutative case. Since α is a k -algebra morphism, we have the following equations

$$(2.7) \quad mc = (1 + dq)D(c) + \{d + \sigma(d) + d\sigma(d)q\}n + c^2q,$$

$$(2.8) \quad md = (1 + dq)\{c + \sigma(c) + D(d)\} + \{d + \sigma(d) + d\sigma(d)q\}m,$$

$$(2.9) \quad dD(r) = \sigma(r)c - cr \quad (r \in R),$$

$$(2.10) \quad d\sigma(r) = \sigma(r)d \quad (r \in R).$$

Assume that S is a right B -Galois extension of R . Then $\gamma_S: S \otimes_R S \rightarrow S \otimes B$ is an epimorphism if and only if there exists elements r_0, r_1, r_2, r_3 in R such that

$$(2.11) \quad r_0 + r_3n = 0,$$

$$(2.12) \quad r_1 + r_2 + r_3m = 0,$$

$$(2.13) \quad r_2c + r_3\{D(c) + \sigma(d)n\} = 1,$$

$$(2.14) \quad r_2d + r_3\{\sigma(c) + \sigma(d)m + D(d)\} = 0.$$

Under these notations, we have the following

Theorem 2.3. *Assume that there exists a right B -Galois extension S/R . If one of the following assumptions satisfies, then B is a Hopf algebra.*

- (1) $c = 0$ and σ is an epimorphism.
- (2) c is not a right zero-divisor in R .
- (3) R has no right zero-divisors.
- (4) $d = p$, or $d = 0$.

Proof. In the following proof, we note that p and q are in the center of R and they are invariant under σ and D .

(1) If $c = 0$, then by (2.13), $r_3\sigma(d)n = 1$. Since σ is an epimorphism, d is in the center of R by (2.10) and so $\sigma(d)$ is invertible in R . Thus by (2.4), $\sigma(d) = p$. By (2.7) and the fact that p and $\sigma(n)$ have left inverse, we get $pq+2 = 0$, which shows that B is a Hopf algebra by (2.2).

(2) If c is not a right zero-divisor, then by (2.3) $d = p$. Using this and (2.8), we have

$$(2.15) \quad (pq+1)\{c + \sigma(c) + pm\} = 0.$$

Since γ_s is a monomorphism, (2.11), (2.12), (2.14) and

$$(2.16) \quad r_2c + r_3\{D(c) + \sigma(d)n\} = 0$$

imply that $r_0 = r_1 = r_2 = r_3 = 0$, and by (2.14) and (2.16), we get that

$$(2.17) \quad pD(c) + p^2n - \sigma(c)c - pmc \text{ is not a right zero-divisor.}$$

Then noting that (2.7), (2.9) and (2.17), we can prove that

$$(2.18) \quad (pq+1)\{p^2n + \sigma(c)c\} \text{ is not a right zero-divisor.}$$

Thus by (2.15), $pm = -c - \sigma(c)$. Multiplying (2.7) by p and using (2.9), we get $(pq+2)\{p^2n + \sigma(c)c\} = 0$ and by (2.18), $pq+2 = 0$.

(3) By (2), we may assume that $c = 0$. If $d = 0$, then α is the identity morphism and S is not a right B -Galois extension of R . If $d \neq 0$, then $d = p$ and by (2.9) $D = 0$. Since γ_s is an epimorphism, $r_3pn = 1$ by (2.13). Moreover by (2.7), $(pq+2)pn = 0$ and so $pq+2 = 0$.

(4) If $d = p$, then the result is clear by the proof of (2). If $d = 0$, then by (2.7) and (2.13), we get $r_2c + r_3D(c) = (r_2 + r_3m - r_3cq)c = 1$, which shows that c has a left inverse element. Thus by (2), B is a Hopf algebra.

In [6], S. Ikehata got some results for an ideal generated by a single polynomial in a skew polynomial ring and in [9], T. Nagahara studied many type of skew polynomials of degree 2 in detail with respect to non-commutative Galois extensions. And there are many papers for quadratic extensions and skew polynomial rings. But in any cases it is not known that there exists a quadratic free right B -Galois extension of R . Because our type of right B -Galois extensions is beyond the scope of the recent work of quadratic extensions and skew polynomial rings.

Question. Does there exist a free quadratic right B -Galois extension of R ?

3. The case of finite cyclic semigroup. Let G be a finite cyclic semigroup (not a group) with identity 1, that is,

$$G = \{1, \sigma, \sigma^2, \dots, \sigma^{n-1} \mid \sigma^n = \sigma^m \text{ for some } 1 \leq m \leq n-1\}.$$

Let $B = kG$ be a semigroup bialgebra with usual coalgebra structure $\Delta(\sigma) = \sigma \otimes \sigma$ and $\varepsilon(\sigma) = 1$, S a commutative right B -Galois extension of k , and $\alpha: S \rightarrow S \otimes B$ a comodule algebra structure morphism. According to the same method in [1, pp. 35–39], we can determine the structure of S as follows: First, we can set that

$$\alpha(s) = \sum_{i=0}^{n-1} s_i \otimes \sigma^i \quad (s, s_i \in S).$$

Using the comodule structure of S , it is easy to see that

$$s = \sum_{i=0}^{n-1} s_i \quad \text{and} \quad \alpha(s_i) = s_i \otimes \sigma^i.$$

We set $S_i = \{s \in S \mid \alpha(s) = s \otimes \sigma^i\}$. Then S_i is a k -module and $S = S_0 \oplus \dots \oplus S_{n-1}$ as a k -module and $S_0 = k$. Since $\gamma_S: S \otimes S \rightarrow S \otimes B$ is an epimorphism, there exist $s_i, t_i \in S_i$ such that

$$\beta_S\left(\sum_{i,j=0}^{n-1} s_i \otimes t_j\right) = \sum_{i,j=0}^{n-1} s_i t_j \otimes \sigma^j = 1 \otimes \sigma.$$

Therefore $(\sum_{i=0}^{n-1} s_i) t_1 = 1$. By $\alpha(s_i t_j) = \alpha(s_i) \alpha(t_j) = s_i t_1 \otimes \sigma^{i+1}$, $s_i t_1$ are contained in S_{i+1} and so $1 \in S_1 \oplus S_2 \oplus \dots \oplus S_{n-1}$, because $S_n = S_m$. This contradicts to $S_0 \cap (S_1 \oplus S_2 \oplus \dots \oplus S_{n-1}) = 0$. Thus there does not exist a right kG -Galois extension of k for a cyclic semigroup G .

Next let B be the dual bialgebra of kG , that is, $B = \text{Hom}(kG, k)$, S/k a right B -extension and $\alpha: S \rightarrow S \otimes B$ a comodule algebra structure morphism. Then by the coalgebra structure of kG , G acts on S as a k -algebra morphism and

$$\alpha(s) = \sum_{i=0}^{n-1} \sigma^i(s) \otimes v_i,$$

where $\{v_0, v_1, \dots, v_{n-1}\}$ are the ordered dual basis of B with respect to the basis $\{1, \sigma, \dots, \sigma^{n-1}\}$ of kG (cf. [8, Example 1]). Assume that S/k is a right B -Galois extension. Then $\gamma_S: S \otimes S \rightarrow S \otimes B$ is an epimorphism and so there exist $s_i, t_i \in S$ such that

$$\gamma_S\left(\sum_{i=0}^r s_i \otimes t_i\right) = \sum_{i=0}^r \sum_{j=0}^{n-1} s_i \sigma^j(t_i) \otimes v_j = 1 \otimes v_0.$$

Therefore

$$\sum_{i=0}^r s_i t_i = 1 \quad \text{and} \quad \sum_{i=0}^r s_i \sigma^j(t_i) = 0$$

for any $j = 1, 2, \dots, n-1$. Since $\sigma^n = \sigma^m$ and σ is a k -algebra morphism, we have the following contradiction :

$$\begin{aligned} 0 &= \sigma^m \left(\sum_{i=0}^r s_i \sigma^{n-m}(t_i) \right) = \sum_{i=0}^r \sigma^m(s_i) \sigma^n(t_i) \\ &= \sum_{i=0}^r \sigma^m(s_i) \sigma^m(t_i) = \sigma \left(\sum_{i=0}^r s_i t_i \right) = \sigma(1) = 1. \end{aligned}$$

Thus S/k is not a right B -Galois extension. These prove the following

Theorem 3.1. *Let G be a finite cyclic semigroup (not a group) and let B be the semigroup bialgebra kG or $(kG)^* = \text{Hom}(kG, k)$. Then there does not exist a right B -Galois extension of k .*

REFERENCES

- [1] S. U. CHASE and S. E. SWEEDLER : Hopf Algebras and Galois Theory, Lecture Notes in Math. 97. Springer-Verlag, Berlin, 1969.
- [2] S. U. CHASE, D. K. HARRISON and A. ROSENBERG : Galois theory and Galois cohomology of commutative rings, Mem. Amer. Math. Soc. 52 (1965), 15–33.
- [3] Y. DOI and M. TAKEUCHI : Cleft comodule algebras for a bialgebra, Comm. Alg. 14 (1986), 801–818.
- [4] Y. DOI and M. TAKEUCHI : Hopf-Galois extension of algebras, the Miyashita-Ulbrich action, and Azumaya algebras, J. Alg. 121 (1989), 488–516.
- [5] J. A. GREEN, W. D. NICHOLS and E. J. TAFT : Left Hopf algebras, J. Alg. 65 (1980), 399–411.
- [6] S. Ikehata : On separable polynomials and Frobenius polynomials in skew polynomial rings, Math. J. Okayama Univ. 22 (1980), 115–129.
- [7] H. F. KREIMER : Quadratic Hopf algebras and Galois extensions, Contemporary Math. 13 (1982), 353–361.
- [8] H. F. KREIMER and M. TAKEUCHI : Hopf algebras and Galois extensions of an algebra, Indiana Univ. Math. J. 30 (1981), 675–692.
- [9] T. NAGAHARA : On separable polynomials of degree 2 in skew polynomial rings, Math. J. Okayama Univ. 19 (1976), 65–95.
- [10] W. D. NICHOLS and E. J. TAFT : The left antipodes of a left Hopf algebra, Contemporary Math. 13 (1982), 363–368.
- [11] M. ORZECZ : Onto endomorphisms are isomorphism. Amer. Math. Monthly 78 (1971), 357–362.
- [12] M. E. SWEEDLER : Hopf Algebras, Benjamin, New York, 1969.

DEPARTMENT OF MATHEMATICS
COLLEGE OF LIBERAL ARTS AND SCIENCES
OKAYAMA UNIVERSITY
2-1-1 TSUSHIMA-NAKA, OKAYAMA 700, JAPAN

(Received July 5, 1990)