Linearly compact dual-bimodules

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LINEARLY COMPACT DUAL-BIMODULES

Dedicated to Professor Kentaro Murata on his 70th birthday

YOSHIKI KURATA and SHIGEYUKI TSUBOI

Let $R$ and $S$ be rings with identity and $_RQ_S$ an $(R, S)$-bimodule. In the previous paper [3], it is shown that if $Q_S$ is quasi-injective and the canonical ring homomorphism $\lambda: R \to \operatorname{End}(Q_S)$ is surjective, then the pair of functors

$$H' = \operatorname{Hom}_R(-, Q) : _RM \to NS$$

and

$$H'' = \operatorname{Hom}_S(-, Q) : NS \to _RM$$

defines a duality between $_RM$ and $NS$, where $_RM$ is the full subcategory of $R$-mod of finitely generated $Q$-torsionless $R$-modules and $NS$ is the full subcategory of $\operatorname{mod}S$ whose objects are all the $S$-modules $N$ such that there exists an exact sequence of the form $0 \to N \to Q^n \to Q'$ for some $n > 0$ and some set $I$.

In this note, we shall give, in the first section, some characterizations of self-cogenerators and then point out that the linearly compactness of a ring is very closed to the existence of some kind of left dual-bimodules. Characterizing these left dual-bimodules, in the second section, we shall show that, for a left dual-bimodule $_RQ_S$ with $Q$ finitely generated, $Q_S$ quasi-injective and $\lambda$ surjective, $Q_S$ is linearly compact if and only if the duality mentioned above can be extended to a duality between $_RFG$ and $\overline{NS}$ (see below for the definition).

1. An $S$-module $Q_S$ will be called a self-cogenerator provided that every right $S$-module isomorphic to a submodule of a factor module of $Q^a, n = 1, 2, \cdots$, is $Q$-torsionless [7, Definition 3.1]. Trivially each cogenerator in $\operatorname{mod}S$ is a self-cogenerator. First, we shall give some characterizations of self-cogenerators. As is easily seen, we have

**Lemma 1.** Let $Q_S$ be an $S$-module. Suppose that

$$0 \to N' \to N \to N'' \to 0$$

is an exact sequence of right $S$-modules such that $N'$ and $N''$ are $Q$-torsionless and $Q$ is $N$-injective. Then $N$ is $Q$-torsionless.

**Lemma 2.** Let $Q_S$ be a quasi-injective $S$-module. Suppose that every
factor module of \( Q \) is \( Q \)-torsionless. Then every factor module of \( Q^n \) is also \( Q \)-torsionless for \( n = 1, 2, \ldots \).

**Proof.** We may show the case where \( n = 2 \). Let \( Q' \) be any submodule of \( Q^2 \) and let \( p : Q^2 \to Q \) be the canonical projection. Then the induced homomorphism \( \bar{p} : Q^2/Q' \to Q/p(Q') \) is an epimorphism with \( \text{Ker} \ \bar{p} \) \( Q \)-torsionless. By assumption \( Q/p(Q') \) is \( Q \)-torsionless and hence \( Q^2/Q' \) is also \( Q \)-torsionless by Lemma 1.

An \((R, S)\)-bimodule \( Q \) will be called a left dual-bimodule provided that \( \ell_R \ell_S(A) = A \) for every left ideal \( A \) of \( R \) and \( r_\ell \ell_S(Q') = Q' \) for every \( S \)-submodule \( Q' \) of \( Q \) (see [3]). A ring that has the double annihilator property ([1, Exercise 24.11]) will be called a dual ring. Hence a dual ring \( R \) is a left dual-bimodule regarded as an \((R, R)\)-bimodule. It is also a right dual-bimodule by defining symmetrically. In [3, Lemma 1.3], it is shown that if \( _S Q_S \) is a left dual-bimodule, then every factor module of \( Q_S \) is \( Q \)-torsionless. Hence we have

**Corollary 3.** Let \( _S Q_S \) be a left dual-bimodule with \( Q_S \) quasi-injective. Then \( Q_S \) is a self-cogenerator.

**Proposition 4.** Let \( _S Q_S \) be an \((R, S)\)-bimodule with \( Q_S \) quasi-injective and \( \lambda \) surjective. Then the following conditions are equivalent:

(1) \( Q_S \) is a self-cogenerator.

(2) Every factor module of \( Q^n \) is \( Q \)-torsionless for \( n = 1, 2, \ldots \).

(3) Every factor module of \( Q_S \) is \( Q \)-torsionless.

(4) \( \ell_R \ell_S Q' = Q' \) for every submodule \( Q' \) of \( Q_S \).

(5) \( N_S = \{| N_S | 0 \to N \to Q^n \text{ is exact for some } n > 0 \} \).

(6) Every submodule of \( Q^n \) is \( Q \)-reflexive for \( n = 1, 2, \ldots \).

**Proof.** (1) \( \Rightarrow \) (2) and (2) \( \Rightarrow \) (3) are evident. The equivalence of (3) and (4) follows from [3, Lemma 1.3] and (3) \( \Rightarrow \) (5) follows from Lemma 2.

(5) \( \Rightarrow \) (6). Let \( N_S \) be a submodule of \( Q^n \). Then \( N \subseteq N_S \) and \( 0 \to N \to Q^n \to Q^m \) is exact for some \( m > 0 \) and \( I \). Since \( Q_S \) is \( Q \)-injective and \( Q \)-reflexive, \( Q \) is \( Q^n \)-injective and \( Q^m \) is \( Q \)-reflexive. Hence by [3, Lemma 3.1], \( N \) must be \( Q \)-reflexive.

(6) \( \Rightarrow \) (1) also follows from [3, Lemma 3.1].

**Remarks.** (1) The equivalence of (1) and (3) of Proposition 4 has already shown in [5, Lemma 1.1].
(2) In case $Q_S$ is a finitely cogenerated cogenerator, then a right $S$-module $N$ is finitely cogenerated if and only if there is an $n > 0$ such that $0 \to N \to Q^n$ is exact by [1, Exercise 10.3]. For example, each dual-bimodule $\mu Q_S$ with $Q_S$ injective and $\lambda$ surjective is a finitely cogenerated cogenerator as an $S$-module by [3, Proposition 1.8 and Lemma 3.5].

**Corollary 5.** For a ring $R$ with $R_R$ injective, the following conditions are equivalent:

1. $R_R$ is a self-cogenerator.
2. Every finitely generated right $R$-module is torsionless.
3. Every cyclic right $R$-module is torsionless.
3'. Every simple right $R$-module is torsionless.
4. $\nu R(\mathfrak{a}) = A$ for every right ideal $A$ of $R$.
5. $\mu R = |N_R| N_R \to N \to R^n$ is exact for some $n > 0$.
5'. $\mu R_R = |N_R| N_R$ is finitely cogenerated.
6. Every submodule of $R^n_R$ is reflexive for $n = 1, 2, \ldots$.
6'. Every finitely cogenerated right $R$-module is reflexive.

**Proof.** (3) $\Rightarrow$ (3') and (3') $\Rightarrow$ (3) are evident and (3') $\Rightarrow$ (3') follows from [1, Proposition 18.15].

Assume (5). Then since (3) and (5) are equivalent, $R_R$ is an injective cogenerator and hence is a finitely cogenerated cogenerator by [6, Satz 3]. Assume (5'). Then since $R$ is in $N_R$, $R_R$ is finitely cogenerated and injective. Hence it is a finitely cogenerated cogenerator again by [6, Satz 3]. Therefore, the equivalence (5) and (5') follows from Remarks (2).

(6) $\Leftrightarrow$ (6') $\Leftrightarrow$ (3') are evident. Hence (6) and (6') are equivalent.

A ring $R$ is a cogenerator ring in case both $\mu R$ and $R_R$ are cogenerators [1, Exercise 24.10].

**Corollary 6.** For a ring $R$ with $R_R$ injective, the following conditions are equivalent:

1. $R$ is a dual ring.
2. $\mu R$ and $R_R$ are self-cogenerators.
3. $R$ is a cogenerator ring.

**Proof.** (1) $\Rightarrow$ (3). As we shall show in Corollary 11, if $R$ is a dual ring, then $R_R$ injective is equivalent to $\mu R$ being injective. Hence, from Corollaries 3 and 5 (1) $\Rightarrow$ (3) follows.
(3) $\Rightarrow$ (2) is evident.

(2) $\Rightarrow$ (1). To prove (1) $\Rightarrow$ (4) of Proposition 4, it is sufficient to assume that $\lambda$ is surjective. Hence, in Corollary 5 (1) $\Rightarrow$ (4) is always valid. Thus (2) implies (1).

Let $N_S$ be an $S$-module, $(x_i)_t$, an indexed set of elements of $N$ and $(N_i)_t$ an indexed set of submodules of $N$. Then the set of congruences $|x \equiv x_i \pmod{N_i}|$ is said to be solvable (finitely solvable), if there is a $y$ in $N(a y_F$ in $N$ for each finite subset $F$ of $I$) such that $y-x_i$ in $N_i$ for each $i$ in $I$. If every finitely solvable set of congruences in $N$ is solvable, then $N$ will be called linearly compact ([7, Definition 2.1]). Using this notion we can characterize left dual-bimodules.

**Proposition 7.** Let $\mathfrak{g}Q_S$ be an $(R, S)$-bimodule with $Q_S$ linearly compact quasi-injective and $\lambda$ an isomorphism. Then the following conditions are equivalent:

1. $Q$ is a left dual-bimodule.
2. $Q_S$ is a self-cogenerator and has essential socle.

Moreover, if this is the case, $\mathfrak{g}Q$ is injective and $\mathfrak{g}R$ is linearly compact.

**Proof.** (1) $\Rightarrow$ (2) follows from Corollary 3 and [3, Proposition 1.8].

(2) $\Rightarrow$ (1). By [7, Lemma 3.7] every cyclic left $R$-module is $Q$-torsionless. Hence by Proposition 4 and [3, Lemma 1.2] $Q$ is a left dual-bimodule.

The last part follows from (2) by [7, Lemmas 3.5 and 3.7].

**Corollary 8.** For a ring $R$ with $R_S$ linearly compact and injective, the following conditions are equivalent:

1. $R$ is a dual ring.
2. $R_S$ is a self-cogenerator and has essential socle.
3. $\mathfrak{g}R$ is a self-cogenerator and has essential socle.

**Proof.** As is remarked in the proof of Proposition 7, (2) implies that $\mathfrak{g}R$ is linearly compact and injective. Hence, again by Proposition 7, (3) is equivalent to $R$ being a dual ring.

As a consequence of Proposition 7 and [7, Theorem 3.10], we have

**Theorem 9.** A ring $R$ is left linearly compact if and only if there exists a left dual-bimodule $\mathfrak{g}Q_S$ such that $Q_S$ is linearly compact quasi-injective and $\lambda$
is surjective.

2. A subcategory of the module category will be called \textit{finitely closed} if it is closed under submodules, factor modules and finite direct sums \cite[p. 465]{4}. Let $\_S Q_S$ be an \((R, S)\)-bimodule. Following \cite{7}, consider the full subcategory of $R\text{-mod}$ consisting of all modules isomorphic to factor modules of submodules of $R^n$ for $n = 1, 2, \ldots$. This is the full subcategory consisting of all modules isomorphic to submodules of factor modules of $R^n$ for $n = 1, 2, \ldots$ and hence is equal to

$$|_{_{R}M}|0 \to M \to M'$$ is exact for some $M' \in _{_{R}}FG|.$

As is easily seen, this is the smallest one of the finitely closed subcategory containing either $R$ or $_{_{R}}FG$. We shall denote this by $_{_{R}}\overline{FG}$, where $_{_{R}}FG$ means the full subcategory of finitely generated left $R$-modules.

Similarly, the full subcategory of $\text{mod-S}$ consisting of all modules isomorphic to factor modules of submodules of $Q^n$ for $n = 1, 2, \ldots$ coincides with one consisting of all modules isomorphic to submodules of factor modules of $Q^n$ for $n = 1, 2, \ldots$. This is the smallest one of the finitely closed subcategory containing either $Q$ or the class of $S$-modules

$$|N_S|0 \to N \to Q^n$$ is exact for some $n > 0|.$

By Proposition 4 this is the smallest one of the finitely closed subcategory containing $N_S$ in case $Q_S$ is a quasi-injective self-cogenerator and $\lambda$ is surjective. Hence we shall denote this by $\overline{N}_S$. Furthermore, $\overline{N}_S$ also coincides with

$$|N_S|N' \to N \to 0$$ is exact for some $N' \in N_S|.$

We are now ready to characterize those left dual-bimodules mentioned in Theorem 9 by means of a duality.

\textbf{Theorem 10.} Let $\_S Q_S$ be a left dual-bimodule with $\_S Q$ finitely generated and $\lambda$ surjective. Then the following conditions are equivalent:

1. $Q_S$ is a linearly compact quasi-injective module.
2. The pair $(H', H')$ defines a duality between $\_S\overline{FG}$ and $\overline{N}_S$.
3. $\_S Q$ is an injective cogenerator.

\textbf{Proof.} (1) $\Rightarrow$ (2) follows from Proposition 7 and \cite[Theorem 3.8]{7}.

(2) $\Leftrightarrow$ (3). Assume (2). Then since $\_S Q \in \_S\overline{FG}$, we can apply \cite[Exercise 20.5]{1} to show that, for each $M \in \_S\overline{FG}$, $\sigma_M$ is an epimorphism by a
similar way as in [1, Theorem 23.5]. In particular, every cyclic $R$-module is $Q$-reflexive by [3, Lemma 1.2] and thus $Q_3$ is quasi-injective by [3, Theorem 3.2]. Furthermore, if $N \subseteq N_3$, then $H'(N)$ is in $\mathfrak{F}\mathfrak{G}$ and hence $\sigma_{H';N}$ is an epimorphism, which shows that $N(\subseteq H'H(N))$ is $Q$-reflexive [1, Proposition 20.14]. In particular, every factor module of $Q_3$ is $Q$-reflexive. Hence, by [2, Theorem 10], $\mathfrak{F}Q$ is an injective cogenerator.

(3) $\iff$ (1) follows from [2, Theorem 10] and [7, Theorem 3.6].

The following corollary follows from Theorem 10 and [3, Lemma 3.5].

**Corollary 11.** For a dual ring $R$, the following conditions are equivalent:

(1) $R_3$ is linearly compact and is injective.
(2) $\mathfrak{F}R$ is linearly compact and is injective.
(3) $\mathfrak{F}R_3$ defines a duality between $\mathfrak{F}\mathfrak{G}$ and $\mathfrak{F}\mathfrak{G}_R$.
(4) $\mathfrak{F}R$ is injective.
(5) $R_3$ is injective.

Finally, we shall remark that a cogenerator ring is also a dual ring satisfying the equivalent condition of Corollary 11 [1, Exercise 24.12].

**References**


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