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NOTE ON THE ISOMORPHISM CLASS GROUPS OF HOPF GALOIS EXTENSIONS

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Let R be a commutative ring with identity and let H be a finite Hopf algebra over R. For a commutative ring extension S/R, the notion of Galois H-object S over R was introduced by S. U. Chase and M. E. Sweedler in [1], and H is called a *Galois Hopf algebra* of S/R. This is a generalization of a separable Galois extension and a purely inseparable extension. If a field K is a Galois extension of a subfield k with Galois group G, then G is uniquely determined. On the other hand, A. Hattori pointed out in [3] that the purely inseparable field extension $K = k[X]/(X^p - r)$ of k of characteristic p has two essentially distinct Galois Hopf algebras $H(0, p)$ and $H(1, p)$ defined below in the sense of Chase and Sweedler [1]. In this note we show that the group of isomorphism classes of Galois objects $\mathrm{Gal}(k, H(0, p))$ and $\mathrm{Gal}(k, H(1, p))$ are isomorphic and give some results with related topics.

In the following, all algebras, morphisms and tensor products are taken over a fixed commutative ring R unless otherwise stated. H is a Hopf algebra which is a finitely generated projective R-module.

Now for the convenience of readers, we review the definitions of Galois objects and related notations according to [1]. A commutative algebra S is called an *H-comodule algebra* if there exists an algebra morphism $\rho_S : S \to S \otimes H$ such that $(\rho_S \otimes I)\rho_S = (I \otimes \Delta)\rho_S$ and $(I \otimes \varepsilon)\rho_S = I$, where I is the identity morphism and Δ, ε are coalgebra structure morphisms of H. For H-comodule algebras S and T with structure morphisms ρ_S and ρ_T respectively, a morphism ϕ: $S \to T$ is called an *H-comodule algebra morphism* if ϕ is an algebra morphism such that $\rho_T\phi = (\phi \otimes I)\rho_S$. S is called a *Galois H-object* over R if $R = S_0 = \{s \in S \mid \rho_S(s) = s \otimes 1\}$, the *invariant subalgebra* of S under ρ_S, S is a faithfully flat R-module and the morphism γ: $S \otimes S \to S \otimes H$ defined by $\gamma(x \otimes y) = (x \otimes 1)\rho_S(y)$ is an isomorphism. Two Galois H-objects S and T are called *isomorphic* if there exists an H-comodule algebra isomorphism ϕ from S to T. Let S and T be Galois H-objects with structure morphisms ρ_S and ρ_T, respectively. Consider the morphism

$$(I \otimes \tau)(\rho_S \otimes I) - I \otimes \rho_T : S \otimes T \to S \otimes T \otimes H,$$

where τ is the twist morphism $x \otimes y \to y \otimes x$. Then the subalgebra $S \cdot T = ker\{(I \otimes \tau)(\rho_S \otimes I) - I \otimes \rho_T\}$ of $S \otimes T$ is a Galois H-object and the H-

comodule structure on $S \cdot T$ is given by $I \otimes \rho_T = (I \otimes \tau)(\rho_S \otimes I)$. Then in the set of isomorphism classes of Galois H-objects $\mathrm{Gal}(R, H)$, we can define the product

$$[S][T] = [S \cdot T] \ ([S], [T] \in \mathrm{Gal}(R, H)),$$

where $[X]$ is the isomorphism class of Galois H-objects which are isomorphic to X, and $\mathrm{Gal}(R, H)$ is an abelian group with identity element $[H]$. These notions are also defined by usual action (cf. [1], [5]).

In the following R is a commutative algebra over the prime field $GF(p)$ $(p \neq 0)$. For an element u in R, we denote by $H(u, p^m)$, the free Hopf algebra over R with basis $\{1, \delta, \cdots, \delta^{p^m-1}\}$ whose Hopf algebra structure is defined as follows :

algebra structure : $\delta^{p^m} = 0$,
coalgebra structure : $\Delta(\delta) = \delta \otimes 1 + 1 \otimes \delta + u(\delta \otimes \delta)$, $\varepsilon(\delta) = 0$,
antipode : $\lambda(\delta) = \sum_{i=1}^{p^m-1}(-1)^i u^{i-1}\delta^i$.

Then in $H(1, p^m)$, if we put $\sigma = \delta + 1$, then $\langle\sigma\rangle$ is a cyclic group of order p^m and $H(1, p^m) = R\langle\sigma\rangle$, where $R\langle\sigma\rangle$ is the group algebra of $\langle\sigma\rangle$. On the other hand, $H(0, p^m)$ is the algebra which is generated by derivation δ of nilpotency index p^m. In general $H(1, p^m)$ and $H(0, p^m)$ are non-isomorphic Hopf algebras.

For an R-algebra $S = R[X]/(X^p - s) = R[x]$ $(s \in R)$, we define a morphism $\rho_S : S \to S \otimes H(0, p)$ by $\rho_S(x) = x \otimes 1 + 1 \otimes \delta$. Then it is easy to check that ρ_S gives an $H(0, p)$-comodule algebra structure on S and S is a Galois $H(0, p)$-object over R (cf. [1, p. 35, Example 4.11]). We set the above type of Galois $H(0, p)$-object by $[x ; s]$. Then we have the following which was proved in [5, Lemma 2.1 and Th. 2.2].

Theorem 1. *Let $S = [x ; s]$ and $T = [y ; t]$ be Galois $H(0, p)$-objects defined as above.*

(1) *Let ϕ: $S \to T$ be a morphism of Galois $H(0, p)$-object. Then ϕ is an isomorphism if and only if there exists an element r in R such that $s - t = r^p$. When this is the case, ϕ is defined by $\phi(x) = y + r$.*

(2) $S \cdot T = [z ; s + t]$.

Proof. (1) By $\rho_T \phi = (\phi \otimes I)\rho_S$, we have $\phi(x) = y + r$ for some r in R. Since ϕ is an algebra morphism, $s - t = r^p$ is clear.

(2) By the definition of the product $S \cdot T$, the subalgebra A of $S \otimes T$

generated by the element $z = x \otimes 1 + 1 \otimes y$ over R is contained in $S \cdot T$ and $z^p = s + t$. Since A is a Galois $H(0, p)$-object in $S \cdot T$, A is equal to $S \cdot T$ by [1, Th. 1.12].

Since R is an algebra over $GF(p)$, $R^p = \{r^p \mid r \in R\}$ is an additive subgroup of the additive group R, and by [5, Th. 1.4], if S is a Galois $H(0, p)$-object over R, then S is isomorphic to $[x ; s]$ for some s in R. Thus we have the following

Corollary 2. Gal$(R, H(0, p))$ *is isomorphic to* R/R^p *as groups.*

Next we consider a Galois $H(1, p)$-object. For $S = R[X]/(X^p - s) = R[x]$, we define an $H(1, p)$-comodule structure on $R[x]$ by $\rho(x) = x \otimes \sigma$. Then by [1, pp. 36−39], $R[x]$ is a Galois $H(1, p)$-object if and only if x^p is invertible in R. We set this type of Galois $H(1, p)$-object by $\langle x ; s \rangle$. Let gal$(R, H(1, p))$ be the set of isomorphism classes of Galois $H(1, p)$-objects $\langle x ; s \rangle$. Then we have the following which is similar to Th. 1 and Cor. 2.

Theorem 3. *Let* $S = \langle x ; s \rangle$ *and* $T = \langle y ; t \rangle$ *be Galois* $H(1, p)$*-objects defined as above.*

(1) *Let* ϕ: $S \to T$ *be a morphism of Galois* $H(1, p)$*-object. Then* ϕ *is an isomorphism if and only if there exists an invertible element* r *in* R *such that* $s = r^p t$. *When this is the case* ϕ *is defined by* $\phi(x) = ry$.

(2) $S \cdot T = \langle z ; st \rangle$.

Proof. (1) By $\rho_T \phi = (\phi \otimes I)\rho_S$, we have $\phi(x) = ry$ for some r in R. Since ϕ is an algebra isomorphism, r is invertible and $s = r^p t$.

(2) It is easy to see that the element $x \otimes y$ in $S \cdot T$ generates a subalgebra A which is a Galois $H(1, p)$-object. Then by [1, Th. 1.12], A is equal to $S \cdot T$.

Corollary 4. gal$(R, H(1, p))$ *is a subgroup of* Gal$(R, H(1, p))$ *and* gal$(R, H(1, p))$ *is isomorphic to* $U(R)/U(R)^p$, *where* $U(R)$ *is the unit group of* R.

In [1, Example 4.16], S. U. Chase proved the following theorem. Let R be an arbitrary commutative ring and let G be a cyclic group of order n. Then there exists a one-to-one correspondence between Galois RG-objects and pairs (I, β), where I is an invertible R-module and β: $I \otimes I \otimes \cdots \otimes I$ (n-times) $\to R$ is an R-module isomorphism. Therefore, gal$(R, H(1, p))$

does not equal $\mathrm{Gal}(R, H(1, p))$ for a certain ring R and if R is a field, $\mathrm{gal}(R, H(1, p))$ equals $\mathrm{Gal}(R, H(1, p))$.

By Cor. 2 and Cor. 4, we have the following

Corollary 5. *Let k be a field of characteristic p. Then the following conditions are equivalent:*

(1) *k is a perfect field.*

(2) $\mathrm{Gal}(k, H(0, p)) = 0$.

(3) $\mathrm{Gal}(k, H(1, p)) = 1$.

In general, we have the following

Theorem 6. *If k is a field of characteristic p, then* $\mathrm{Gal}(k, H(0, p))$ *is isomorphic to* $\mathrm{Gal}(k, H(1, p))$ *as groups.*

Proof. Let k be an infinite field and let K be an extension field of k. First we show that $\#k \leq \#(U(K)/U(k))$, where $\#X$ is the cardinality of X. Let x be an element in K which does not contained in k. For elements a, b in k, we assume that $U(k)(x+a) = U(k)(x+b)$. Then there exists an element c in $U(k)$ such that $x+a = c(x+b)$ and so $(1-c)x+(a-cb) = 0$. Since $1-c$ and $a-cb$ are contained in k, we have $c = 1$ and $a = cb$. Therefore $a = b$ and thus $\#k \leq \#(U(K)/U(k))$. Now in the proof of the theorem, we may assume that $k \neq k^p$. Since k/k^p and $U(k)/U(k)^p$ are elementary abelian p-groups, it suffices to show that $\#(k/k^p) = \#(U(k)/U(k)^p)$. As vector spaces over k^p, we have $\#k^p \leq \#(k/k^p) \leq \#k$. But since k is isomorphic to k^p and the fact we have just shown above, $\#k = \#(k/k^p) = \#(U(k)/U(k)^p)$.

For a separable field extension, we have the following example which was given in [5, Remark 2].

Example 7. Let k be the prime field $GF(2)$. Then the polynomial X^4+X+1 is separable irreducible in $k[X]$ and so $K = k[X]/(X^4+X+1)$ is a cyclic 2^2-extension of k with Galois group $\langle\sigma\rangle$ of order 4. Thus K is a Galois $k\langle\sigma\rangle^*$-object over k, where $k\langle\sigma\rangle^* = \mathrm{Hom}_k(k\langle\sigma\rangle, k)$ is the dual Hopf algebra of the group algebra $k\langle\sigma\rangle$. On the other hand, let H be a free k-module with basis $\{1, D, D^2, D^3\}$. The Hopf algebra structure of H is defined by $D^4 = D$, $\Delta(D) = D \otimes 1 + 1 \otimes D$, $\varepsilon(D) = 0$ and $\lambda(D) = -D$. Then by [5, Th. 1.3], K is a Galois H-object of k and we can see that $z^2 = 0$ or

$z^2 = 1$ for any $z \in H^*$. Thus $k\langle\sigma\rangle$ is not isomorphic to H^* as Hopf algebras. This shows that $K = k[X]/(X^4+X+1)$ has two non-isomorphic Galois Hopf algebras $k\langle\sigma\rangle^*$ and H.

For the above Hopf algebras $R\langle\sigma\rangle^*$ and H, the isomorphism class groups $\mathrm{Gal}(R, R\langle\sigma\rangle^*)$ and $\mathrm{Gal}(R, H)$ were also computed for an arbitrary commutative algebra R over $GF(2)$. Since $R\langle\sigma\rangle = H(1, 2^2)$, then by [4, Th. 3.2.4], there is a group isomorphism

$$\mathrm{Gal}(R, R\langle\sigma\rangle^*) \cong R_2^+/M_1,$$

where $R_2^+ = R\times R$, the cartesian product of R with addition defined by

$$(s_1, t_1)+(s_2, t_2) = (s_1+s_2, s_1s_2+t_1+t_2)$$

and $M_1 = \{(r^2+r, r(r^2+r)+s(1+s)) \mid r, s \in R\}$. On the other hand, by [5, Th. 2.2], there is a group isomorphism

$$\mathrm{Gal}(R, H) \cong R/\{r^4+r \mid r \in R\}.$$

If we take $R = GF(2)$, then $M_1 = (0, 0)$ and $\{r^4+r \mid r \in R\} = 0$ and so $\mathrm{Gal}(GF(2), GF(2)\langle\sigma\rangle^*) \cong GF(2)\times GF(2)$ which is a cyclic group of order 4 by definition of addition, and $\mathrm{Gal}(GF(2), H) \cong GF(2)$. Therefore

Theorem 7. *Under the above notations*, $\mathrm{Gal}(GF(2), GF(2)\langle\sigma\rangle^*)$ *is not isomorphic to* $\mathrm{Gal}(GF(2), H)$.

For a separable field extension with characteristic 0, the similar example was obtained in [2, Example 2.3] and they showed that for the rational number field Q, the field extension $Q[\sqrt[4]{2}]/Q$ has two different type of Galois Hopf algebras H_1 and H_2. But it is not known that the isomorphism class groups $\mathrm{Gal}(Q, H_1)$ and $\mathrm{Gal}(Q, H_2)$ are isomorphic or not.

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