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## On Real Hypersurfaces in a Complex Space Form

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# ON REAL HYPERSURFACES IN A COMPLEX SPACE FORM

HIROYUKI KURIHARA

## 0. INTRODUCTION

A complex  $n$ -dimensional Kähler manifold of constant holomorphic sectional curvature  $4c$  is called a complex space form, which denoted by  $M_n(c)$ . The complete and simply connected complex space form is a complex projective space  $P_n(\mathbb{C})$ , a complex Euclidean space  $\mathbb{C}$ , or a complex hyperbolic space  $H_n(\mathbb{C})$ , according as  $c > 0, c = 0, c < 0$ .

The induced almost contact metric structure of a real hypersurface  $M$  in  $M_n(c)$  is denoted by  $(\phi, \xi, \eta, \langle \cdot, \cdot \rangle)$ .

Typical examples of real hypersurfaces in  $P_n(\mathbb{C})$  are homogeneous one. R.Takagi ([18]) classified all homogeneous real hypersurfaces in  $P_n(\mathbb{C})$  into six types. Namely he proved

**Theorem A** ([18]). *Every homogeneous real hypersurfaces in  $P_n(\mathbb{C})$  is locally congruent to one of the following;*

- (A<sub>1</sub>) a tube over a hyperplane  $P_{n-1}(\mathbb{C})$ ,
- (A<sub>2</sub>) a tube over a totally geodesic  $P_k(\mathbb{C}) (1 \leq k \leq n-2)$ ,
- (B) a tube over a complex quadric  $Q_{n-1}$ ,
- (C) a tube over a  $P_1(\mathbb{C}) \times P_{(n-1)/2}(\mathbb{C})$  and  $n$  is odd,
- (D) a tube over a complex Grassmann  $G_{2,5}(\mathbb{C})$  and  $n = 9$ ,
- (E) a tube over a Hermitian symmetric space  $SO(10)/U(5)$  and  $n = 15$ .

On the other hand, J.Berndt([2]) classified all real hypersurfaces in  $H_n(\mathbb{C})$  with constant principal curvatures under the condition such that  $\xi$  is principal. Namely he proved

**Theorem B** ([2]). *Let  $M$  be real hypersurfaces in  $H_n(\mathbb{C})$ . Then  $M$  has constant principal curvature and  $\xi$  is principal if and only if  $M$  is locally congruent to one of the following;*

- (A<sub>0</sub>) a horosphere,
- (A<sub>1</sub>) a geodesic hypersphere or a tube over a hyperplane  $H_{n-1}(\mathbb{C})$ ,
- (A<sub>2</sub>) a tube over a totally geodesic  $H_k(\mathbb{C}) (1 \leq k \leq n-2)$ ,
- (B) a tube over a totally real hyperbolic space  $H_n(\mathbb{R})$ .

Let  $M$  be a real hypersurface of type  $A_1$  or  $A_2$  in  $P_n(\mathbb{C})$  or that of  $A_0$ ,  $A_1$  or  $A_2$  in  $H_n(\mathbb{C})$ . Then  $M$  is said to be of type  $A$  for simplicity.

Moreover, M. Kimura and S. Maeda ([9]) and S. S. Ahn, S. B. Lee and Y. J. Suh ([1]) found non-homogeneous real hypersurfaces in  $P_n(\mathbb{C})$  and  $H_n(\mathbb{C})$ , respectively which is called ruled real hypersurfaces.

The purpose of this paper is to give some characterizations of real hypersurfaces in  $M_n(c)$ ,  $c \neq 0$ . In section 1 we investigate  $M$  by using the action  $\phi$  on the curvature tensor  $R$ . The action of the derivation  $R(X, Y)$  on the algebra of tensor fields of a real hypersurfaces on  $M_n(c)$ ,  $c \neq 0$  have been studied by many authors ([4], [5], [6], [9]~[11] and so on). In particular, we investigate  $M$  by using the action  $R(X, Y)$  on the tensor field  $\phi$  in section 2.

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## 1. PRELIMINARIES

Let  $M$  be a real hypersurface of  $M_n(c)$ ,  $c \neq 0$ . In a neighborhood of each point, we choose a unite normal vector field  $N$  in  $M_n(c)$ . The Levi-Civita connection  $D$  in  $M_n(c)$  and  $\nabla$  in  $M$  are related by the following formulas for any  $X, Y \in \mathfrak{X}(M)$ , where  $\mathfrak{X}(M)$  is the set of all vector fields on  $M$ :

$$(1.1) \quad \begin{aligned} D_X Y &= \nabla_X Y + \langle AX, Y \rangle N, \\ D_X N &= -AX, \end{aligned}$$

where  $\langle, \rangle$  denotes the Riemannian metric of  $M$  induced from the metric  $g$  on  $M_n(c)$  and  $A$  is the shape operator of  $M$ . An eigenvector field  $X$  of the shape operator  $A$  is called a principal curvature vector field. Also an eigenvalue  $\lambda$  of  $A$  called a principal curvature. In what follows, we denote  $V_\lambda$  the eigenspace of  $A$  with eigenvalue  $\lambda$ .

It is known that  $M$  has an almost contact metric structure induced from the complex structure  $J$  on  $M_n(c)$ , i.e., we define a tensor  $\phi$  of type  $(1,1)$ , a vector field  $\xi$  and a 1-form  $\eta$  on  $M$  by the following,

$$(1.2) \quad \langle \phi X, Y \rangle = g(JX, Y), \quad \langle \xi, X \rangle = \eta(X) = g(JX, N).$$

Then we have

$$(1.3) \quad \phi^2 X = -X + \eta(X)\xi, \quad \langle \xi, \xi \rangle = 1, \quad \phi\xi = 0.$$

From (1.1), we have easily

$$(1.4) \quad (\nabla_X \phi)Y = \eta(Y)AX - \langle AX, Y \rangle \xi,$$

$$(1.5) \quad \nabla_X \xi = \phi AX.$$

Let  $R$  be the curvature tensor of  $M$ . Since the curvature tensor of  $M_n(c)$  has a nice form, we have the following Gauss and Codazzi equations:

$$(1.6) \quad \begin{aligned} R(X, Y)Z &= c(\langle Y, Z \rangle X - \langle X, Z \rangle Y \\ &+ \langle \phi Y, Z \rangle \phi X - \langle \phi X, Z \rangle \phi Y - 2\langle \phi X, Y \rangle \phi Z) \\ &+ \langle AY, Z \rangle AX - \langle AX, Z \rangle AY, \end{aligned}$$

$$(1.7) \quad (\nabla_X A)Y - (\nabla_Y A)X = \eta(X)\phi Y - \eta(Y)\phi X - 2\langle \phi X, Y \rangle \xi.$$

## 2. THE ACTION $\phi$ ON THE CURVATURE TENSOR OF $M$ IN $M_n(c), c \neq 0$ .

First of all, for a ruled real hypersurface in  $M_n(c), c \neq 0$ , it is known that following Lemma

**Lemma 2.1** ([1],[9]). *Let  $M$  be a real hypersurface in  $M_n(c), c \neq 0$ . The shape operator  $A$  of  $M$  is written by*

$$(2.1) \quad \begin{aligned} A\xi &= \alpha\xi + \beta W \quad (\beta \neq 0) \\ AW &= \beta\xi \\ AX &= 0 \quad (\text{for any } X \perp \xi, W) \end{aligned}$$

where  $W$  is a unit vector field orthogonal to  $\xi$ ,  $\alpha, \beta \in C^\infty(M)$  ( $C^\infty(M)$  is the set of all smooth functions on  $M$ ) if and only if  $M$  is locally congruent to ruled real hypersurfaces.

Let  $M$  be a real hypersurface in  $M_n(c) (c \neq 0, n \geq 3)$  and  $T_0$  the distribution defined by  $T_0(x) = \{X(x) \in T_x M | X(x) \perp \xi(x)\}$  for  $x \in M$ .

Using (1.3) and (1.6), we get

$$(2.2) \quad \begin{aligned} (\phi R)(X, Y)Z &= \phi R(X, Y)Z - R(X, Y)\phi Z - R(\phi X, Y)Z - R(X, \phi Y)Z \\ &= \langle AY, Z \rangle (\phi A - A\phi)X - \langle AX, Z \rangle (\phi A - A\phi)Y \\ &\quad + \langle (\phi A - A\phi)Y, Z \rangle AX - \langle (\phi A - A\phi)X, Z \rangle AY, \end{aligned}$$

for any  $X, Y, Z \in T_0$ .

The purpose of this section is to prove the following

**Theorem 1.** *Let  $M$  be a real hypersurface in  $M_n(c), (c \neq 0, n \geq 3)$ . Suppose*

$$(2.3) \quad (\phi R)(X, Y)Z = 0,$$

for any  $X, Y, Z \in T_0$ , then  $M$  is locally congruent to be of type A or ruled real hypersurfaces.

Let us prove Theorem 1.

First we assume that the structure vector field  $\xi$  is not principal. Then we can put  $A\xi = \alpha\xi + \beta W$ , where  $W \in T_0, \|W\| = 1$  and  $\alpha, \beta (\neq 0) \in C^\infty(M)$ . From (1.2), (1.3) and (2.2), the equation (2.3) shows

$$\langle (\phi R)(\phi W, X)W - (\phi R)(W, X)\phi W, \xi \rangle = \beta \langle \phi X, A\phi W \rangle = 0,$$

for any  $X \in T_0$ . Therefore, since (1.2) and  $\beta \neq 0$ , we obtain  $\langle \phi A\phi W, X \rangle = 0$ . Together with  $\langle \phi A\phi W, \xi \rangle = 0$ , we find  $\phi A\phi W = 0$ . Applying  $\phi$  to this equality and using (1.3), we get

$$(2.4) \quad A\phi W = 0.$$

Again putting  $X = \phi W$  and  $Z = W$  in (2.3), from (1.3), (2.1) and (2.4) we get

$$(2.5) \quad \langle AY, W \rangle AW - \langle AW, W \rangle AY = 0.$$

Taking the inner product with  $\xi$ , by  $\beta \neq 0$  we have

$$\langle AY, W \rangle = \langle AW, W \rangle \langle Y, W \rangle.$$

Thus we observe  $\langle AW, X \rangle = 0$  for any  $X \in T_0, X \perp W$ . From this equation we can write

$$(2.6) \quad AW = \beta \xi + \gamma W,$$

where  $\gamma = \langle AW, W \rangle \in C^\infty(M)$ .

Putting  $Y = W$  and  $Z = X$  in (2.3) and taking the inner product with  $\xi$ , together with (1.2), (1.3), (2.2), (2.4) and  $\beta \neq 0$ , we observe

$$\langle A\phi X, X \rangle = 0.$$

Replacing  $X$  with  $Y + Z$ , we obtain

$$(2.7) \quad \langle A\phi Y - \phi AY, Z \rangle = 0,$$

for any  $Y, Z \in T_0$ . Again replacing  $Y$  with  $\phi Y$  in above equation, we get

$$(2.8) \quad \langle AY + \phi A\phi Y, Z \rangle = 0.$$

Using (2.2) and (2.7), for any  $X, Y, Z \in T_0$  and  $X, Y, Z \perp W$ , we rewrite (2.3) as

$$\langle AY, Z \rangle (\phi A - A\phi)X - \langle AX, Z \rangle (\phi A - A\phi)Y = 0.$$

Putting  $X = \phi W$  in above equation, by (2.4) and  $\beta \neq 0$ , we obtain

$$\langle AY, Z \rangle = 0.$$

Above equation implies

$$(2.9) \quad \gamma = 0 \text{ and } AY = 0,$$

for any  $Y \perp \xi, W$ . Thus because of (2.6), (2.9) and Lemma 2.1,  $M$  is locally congruent to a ruled real hypersurface.

Next we assume that the structure vector field  $\xi$  is principal with corresponding principal curvature  $\alpha$ . Then the following three propositions are known.

**Proposition A** ([6], [12]). *If  $\xi$  is a principal curvature vector field, then the corresponding principal curvature  $\alpha$  is locally constant.*

**Proposition B** ([12]). *If  $\xi$  is a principal curvature vector field with corresponding principal curvature  $\alpha$ . Suppose  $X_\lambda \in V_{\lambda(\neq \alpha)}$  and  $\alpha \neq 2\lambda$ , then  $\phi X_\lambda \in V_{\mu(=(\alpha\lambda+2c)/(2\lambda-\alpha))}$ .*

**Proposition C** ([12]). *If  $\xi$  is a principal curvature vector field with corresponding principal curvature  $\alpha$ . Suppose  $\phi X_\lambda \in V_\lambda$  for any vector fields  $X_\lambda \in V_{\lambda(\neq \alpha)}$ . Then  $\phi$  and  $A$  are commutative.*

Furthermore, M.Okumura([14]) and S.Montiel and A.Romero([13]) proved

**Theorem C** ([13],[14]). *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c \neq 0$ . Then  $\phi$  and  $A$  are commutative if and only if  $M$  is locally congruent to be of type A real hypersurfaces.*

We can consider the following two cases.

First we suppose that  $\alpha^2 - 4c = 0$ . In this case  $M$  is of type  $A_0$  in Theorem B (for detail see ([2])).

Second we suppose that  $\alpha^2 - 4c \neq 0$ .

**Lemma 2.2.** *Let  $M$  be a real hypersurface in  $M_n(c)$  ( $c \neq 0, n \geq 3$ ) satisfying (2.3) and  $\xi$  is a principal vector field with corresponding principal curvature  $\alpha$ . Suppose that  $\alpha^2 - 4c \neq 0$ . Then there exists a principal curvature vector field  $X_\lambda \in V_{\lambda(\neq \alpha)}$  such that  $\phi X_\lambda \in V_\lambda$ .*

*Proof.* Take an orthonormal frame field  $\{\xi, X_{\lambda_i}, \phi X_{\lambda_i} (i = 1, \dots, n-1)\}$  consisting of principal curvatures by  $\alpha, \lambda_i, \mu_i$ , respectively because of Proposition B. Such a frame field is said to be a *local CR-frame field* on  $M$ . Suppose that  $\lambda_i \neq \mu_i$  for all  $i = 1, \dots, n-1$ . In (2.3) setting  $Y = \phi X_{\lambda_i}$  and  $Z = X_{\lambda_i}$ , using (1.2), (1.3), (2.2) and Proposition B we have

$$AX = \lambda_i \langle X_{\lambda_i}, X \rangle X_{\lambda_i} + \mu_i \langle \phi X_{\lambda_i}, X \rangle \phi X_{\lambda_i}.$$

It follows that  $\lambda_i = \mu_i = 0$ , which is a contradiction.  $\square$

**Lemma 2.3.** *Under the assumptions of Lemma 2.2, the principal curvature of  $\phi X_{\lambda_i}$  is equal to that of  $X_{\lambda_i}$  ( $i = 1, \dots, n-1$ ).*

*Proof.* There exists  $X_\nu \in \{X_{\lambda_i} (i = 1, \dots, n-1)\}$  such that  $\nu^2 = 1$  because of Lemma 2.2 and Proposition B. Then from (2.2) and Proposition B, we get

$$(\phi R)(X_{\lambda_i}, X_\nu)Z = \nu(\lambda_i - \mu_i)(-\langle X_\nu, Z \rangle \phi X_{\lambda_i} + \langle \phi X_{\lambda_i}, Z \rangle X_\nu) = 0,$$

for any  $Z \in T_0$ . It follows that  $\lambda_i = \mu_i$ .  $\square$

Hence from Proposition C and Theorem C,  $M$  is locally congruent to be of type  $A_1$  or  $A_2$  in Theorem A. Conversely by Lemma 2.1, the shape operator  $A$  of ruled real hypersurfaces  $M$  in  $M_n(c)$  is written by (2.1). From this and Theorem C, it is easily checked that ruled real hypersurfaces and real hypersurfaces of type A satisfy condition (2.3).

It completes the proof of Theorem 1.

**Corollary 2.** *Let  $M$  be a real hypersurface in  $M_n(c)$ , ( $c \neq 0, n \geq 3$ ). If  $(\phi R)(X, Y)Z = 0$  for one of  $X, Y, Z \in \mathfrak{X}(M)$  and the rest in  $T_0$ , then  $M$  is locally congruent to be of type A real hypersurfaces.*

In fact, by (2.1) it is easily checked that ruled real hypersurfaces don't satisfy above condition.

**Remark 2.1.** *Y. Maeda ([12]) classified real hypersurfaces in  $P_n(\mathbb{C})$  ( $n \geq 3$ ) which satisfy the condition  $\phi R \equiv 0$  and  $\xi$  is a principal curvature vector field.*

### 3. CURVATURE OPERATOR OF $M$ IN $M_n(c)$ , $c \neq 0$ .

In this section, we consider the action of the derivation  $R(X, Y)$  on the algebras of tensor fields of a real hypersurface in  $M_n(c)$ ,  $c \neq 0$ . We recall that if  $T$  is a tensor field of type  $(r, s)$ , then  $R(X, Y) \cdot T = \nabla_X \nabla_Y T - \nabla_Y \nabla_X T - \nabla_{[X, Y]} T$  for  $X, Y \in \mathfrak{X}(M)$ .

For brevity of notation, we denote by  $RT$  the tensor of type  $(r, s+2)$  defined by

$$(RT)(X_1, X_2, \dots, X_s, X, Y) = (R(X, Y) \cdot T)(X_1, X_2, \dots, X_s).$$

First of all we shall study real hypersurfaces  $M$  in  $M_n(c)$ ,  $c \neq 0$  which satisfy  $\mathfrak{S}(R\phi)(X, Y)Z = 0$  for any  $X, Y, Z \in \mathfrak{X}(M)$ , where  $\mathfrak{S}$  denotes the cyclic sum with respect to  $X, Y$  and  $Z$ .

Using (1.2), (1.3) and (1.6), we get

$$\begin{aligned} (R\phi)(X, Y)Z &= R(X, Y)\phi Z - \phi R(X, Y)Z \\ (3.1) \quad &= c(\langle \phi X, Z \rangle \eta(Y)\xi - \langle \phi Y, Z \rangle \eta(X)\xi + \eta(X)\eta(Z)\phi Y \\ &\quad - \eta(Y)\eta(Z)\phi X) - \langle AY, Z \rangle \phi AX + \langle AX, Z \rangle \phi AY \\ &\quad + \langle AY, \phi Z \rangle AX - \langle AX, \phi Z \rangle AY, \end{aligned}$$

for any  $X, Y, Z \in \mathfrak{X}(M)$ .

Furthermore by (3.1) we have

$$\begin{aligned} \mathfrak{S}(R\phi)(X, Y)Z &= (R\phi)(X, Y)Z + (R\phi)(Y, Z)X + (R\phi)(Z, X)Y \\ (3.2) \quad &= 2c(\langle X, \phi Y \rangle \eta(Z) + \langle Y, \phi Z \rangle \eta(X) + \langle Z, \phi X \rangle \eta(Y)) \\ &\quad + \langle (A\phi + \phi A)X, Y \rangle AZ + \langle (A\phi + \phi A)Y, Z \rangle AX \\ &\quad + \langle (A\phi + \phi A)Z, X \rangle AY. \end{aligned}$$

**Lemma 3.1.** *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c \neq 0$ . If*

$$(3.3) \quad \mathfrak{S}(R\phi)(X, Y)Z = 0,$$

*for any  $X, Y, Z \in \mathfrak{X}(M)$ , then the structure vector field  $\xi$  is principal.*

*Proof.* We assume that the structure vector field  $\xi$  is not principal. Then we can put  $A\xi = \alpha\xi + \beta W$ , where  $W \in T_0$ ,  $\|W\| = 1$  and  $\alpha, \beta (\neq 0) \in C^\infty(M)$ .

Putting  $X = \xi, Y = W$  and  $Z = \phi W$  in (3.3), by (1.2), (1.3) and (3.2), we have

$$(3.4) \quad -2c\xi + \langle (A\phi + \phi A)\phi W, W \rangle A\xi + \beta AW = 0.$$

From (3.4), we can write

$$(3.5) \quad AW = \beta\xi + \gamma W,$$

where  $\gamma = \langle AW, W \rangle \in C^\infty(M)$ . From (3.4) and (3.5), we obtain  $\langle A\phi W, \phi W \rangle = 0$  and therefore

$$(3.6) \quad \beta^2 = \alpha\gamma - 2c.$$

On the other hand, let  $e_1, \dots, e_{2n-1}$  be a local orthonormal frame field on  $M$ . Let  $X = e_i$  and  $Y = \phi e_i$  in (3.3). Then contraction yields that

$$(3.7) \quad -4c(n-1)\eta(Z)\xi + 2A^2Z - 2A\phi A\phi Z + (1 - \alpha^2 \text{tr} A)AZ = 0.$$

Putting  $Z = \xi$  in (3.7), we find

$$-4c(n-1) + 2\beta^2 - 2\alpha\gamma = 0,$$

together with (3.6), we get  $-4cn = 0$ , which is a contradiction.  $\square$

Suppose  $M$  satisfies (3.3). Then by Lemma 3.1, the structure vector field  $\xi$  is principal with corresponding principal curvature  $\alpha$ .

If  $\alpha^2 - 4c = 0$ , then by the same discussion as in section 2,  $M$  is of type  $A_0$  in Theorem B.

Hence we suppose that  $\alpha^2 - 4c \neq 0$ . Then putting  $X = \xi$  and  $Z = X_\lambda \in V_\lambda$  in (3.2), from (1.2) and (1.3), we find

$$2c + \alpha(\lambda + \mu) = 0.$$

This equation tells us that  $\alpha \neq 0$ . Together with Proposition B, we have

$$\lambda(\alpha\lambda + 2c) = 0.$$

Thus  $M$  has three distinct constant principal curvatures

$$(3.8) \quad 0, \alpha, -\frac{2c}{\alpha}.$$

On the other hand, M.Kimura([8]) proved

**Theorem D ([8]).** *Let  $M$  be a real hypersurface in  $P_n(\mathbb{C})$ . Then  $M$  has constant principal curvature and  $\xi$  is principal if and only if  $M$  is locally congruent to a homogeneous real hypersurface.*

Together with Theorem B,  $M$  is of type  $A_1 \sim E$  in Theorem A or type  $A_0 \sim B$  in Theorem B.

Conversely it is easily checked that these real hypersurfaces don't satisfy (3.8) and therefore also condition (3.3). Thus we obtain following Proposition

**Proposition 3.** *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c \neq 0$ . Then  $M$  cannot satisfy  $\mathfrak{S}(R\phi)(X, Y)Z = 0$  for any  $X, Y, Z \in \mathfrak{X}(M)$ .*

**Remark 3.1.** *R.Takagi([19]) and J.Saito([16]) classified real hypersurfaces in  $P_n(\mathbb{C})$ ,  $n \geq 3$  and  $H_n(\mathbb{C})$ ,  $n \geq 3$ , with three distinct principal curvature, respectively.*

**Corollary 4.** *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c \neq 0$ . Then  $M$  cannot satisfy  $R\phi \equiv 0$ .*

Next, we shall study real hypersurfaces  $M$  in  $M_n(c)$ ,  $c \neq 0$  which satisfy  $(R\phi)(X, Y)Z = 0$  for any  $X, Y, Z \in T_0$ .

Using (3.1), we get

$$(3.9) \quad (R\phi)(X, Y)Z = -\langle AY, Z \rangle \phi AX + \langle AX, Z \rangle \phi AY \\ + \langle AY, \phi Z \rangle AX - \langle AX, \phi Z \rangle AY,$$

for any  $X, Y, Z \in T_0$ . Assume

$$(3.10) \quad (R\phi)(X, Y)Z = 0.$$

First we assume that the structure vector field  $\xi$  is not principal. Then we can put  $A\xi = \alpha\xi + \beta W$ , where  $W \in T_0$ ,  $\|W\| = 1$  and  $\alpha, \beta (\neq 0) \in C^\infty(M)$ .

Taking the inner product of (3.9) with  $\xi$ , from (1.2), (1.3) and (3.10), we have

$$(3.11) \quad \langle \langle Y, W \rangle \phi AX - \langle X, W \rangle \phi AY, Z \rangle = 0.$$

Putting  $X = W$  and  $Z = \phi W$  in (3.11), we observe

$$(3.12) \quad AW = \beta\xi + \gamma W,$$

where  $\gamma = \langle AW, W \rangle \in C^\infty(M)$ . The equation (3.11) also implies

$$\langle Y, W \rangle AX = \langle X, W \rangle AY.$$

This equation tells us that

$$(3.13) \quad AX = 0,$$

for all  $X \in T_0$ ,  $X \perp W$ . In this case, three equations (3.12), (3.13) and  $A\xi = \alpha\xi + \beta W$  imply the type number of  $M$  is smaller than 3, where the *type number* of  $M$  is defined as the rank of  $A$ . For the problem with respect to the type number  $t$ , Y.J.Suh([17]) and M.Ortega and J.D.Perez([15]) showed that

**Theorem E** ([15],[17]). *Let  $M$  be a real hypersurface in  $M_n(c)$  ( $c \neq 0$ ,  $n \geq 3$ ) satisfying  $t(p) \leq 2$  for any point  $p$  in  $M$ . Then  $M$  is a ruled real hypersurface.*

Thus in this case,  $M$  is locally congruent to a ruled real hypersurface.

Second we assume that the structure vector field  $\xi$  is principal with corresponding principal curvature  $\alpha$ . If  $\alpha^2 - 4c = 0$ , then  $M$  is of type  $A_0$  in Theorem B. Thus we suppose that  $\alpha^2 - 4c \neq 0$ . Take a local CR-frame field  $\{\xi, X_{\lambda_i}, \phi X_{\lambda_i} (i = 1, \dots, n-1)\}$  consisting of principal curvatures by  $\alpha, \lambda_i, \mu_i$ , respectively. Putting  $X = Z = X_{\lambda_j}$  and  $Y = X_{\lambda_j}$  in (3.10), by (1.2), (1.3) and (3.9), we obtain  $\lambda_i = 0$  for some  $i$ . Then from Proposition B, we get  $\mu_i = -2c/\alpha$ . Furthermore putting  $X = Z = \phi X_{\lambda_j}$  and  $Y = \phi X_{\lambda_j}$  in (3.10), we have  $\mu_j = 0$  for any  $j \neq i$ . Thus we get  $\lambda_j = -2c/\alpha$  for any  $j \neq i$ . Therefore  $M$  is of type  $A_1 \sim E$  in Theorem A or type  $A_0 \sim B$  in Theorem B.

Conversely it is trivial that these real hypersurfaces don't satisfy (3.8) and from (2.1) ruled real hypersurfaces satisfy (3.10).

Consequently, we obtain the following theorem.

**Theorem 5.** *Let  $M$  be a real hypersurface in  $M_n(c)$ ,  $c \neq 0$ ,  $n \geq 3$ . Suppose  $(R\phi)(X, Y)Z = 0$  for any  $X, Y, Z \in T_0$ , then  $M$  is locally congruent to a ruled real hypersurface.*

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