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CONTINUITY OF ADDITIVE FUNCTIONALS

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1. Let E be a measure space with positive finite measure μ , and S be the space of all essentially finite measurable functions on E . A set X of S will be called *normal* if $x \in X$, $|x| \geq |y|$ imply $y \in X$.

In the following, we assume X is a normal sublattice of S . A functional T defined on X is called additive if $x, y \in X$; $|x| \cap |y| = 0$ imply $T(x+y) = T(x) + T(y)$.

For the integral representation of T , the continuity condition is important.

Recently, L. Drewnowski and W. Orlicz have proved that if T is (uc_∞) and (uac) , then T can be represented by $f(r, t)$ such that

$$T(x) = \int_E f(x(t), t) d\mu$$

where $f(0, t) = 0$ a. e., $f(r, t)$ satisfies the generalized Carathéodory conditions and $f(x(t), t)$ is integrable on E for each $x \in X$. [1]

In this note, we shall prove that the continuity conditions of Drewnowski and Orlicz (cf. also Friedman and Katz [2]) is replaced by apparently weaker condition: order continuity. Here we do not assume any "uniform" property.

The proof of Theorem 7 in [4] is not sufficient, so the assumption of Theorem must be replaced by order-continuity instead of (CIII) (semi-continuity).

A sequence $x_n \in X (n=1, 2, \dots)$ is *order-convergent* to $x \in X$, if there exists $y \in X$, $|x_n| \leq y$ and $x_n(t) \rightarrow x(t)$ a. e. in E .

2. For an additive functional T on X , we shall consider the following conditions:

(uc_∞) T is *uniformly continuous* by $L_\infty(\mu)$ -norm (essential supremum norm) on every set $\{y; |y| \leq |x|\}$ with $x \in X$, $\|x\|_\infty < \infty$.

(uac) T is *uniformly absolutely continuous* on every set $\{y; |y| \leq |x|\}$, $x \in X$; that is for each $\varepsilon > 0$, there is a positive number $\delta > 0$ such that $|T(y\chi_e)| \leq \varepsilon$ for $|y| \leq |x|$ and a measurable set e with $\mu(e) < \delta$ where χ_e is a characteristic function of e .

(o) If $x_n \in X (n=1, 2, \dots)$ is order-convergent to $x \in X$, then $\lim_{n \rightarrow \infty} T(x_n) = T(x)$.

Theorem (o) is equivalent to (uc_∞) and (uac) .

Proof. Since (uc_∞) and (uac) imply (o) obviously, we shall prove the converse.

(o) $\Rightarrow$ (uac) This can be proved by the same method used in the proof of theorem 1.3 of Drewnowski and Orlicz [1].

(o) $\Rightarrow$ (uc_∞) It is sufficient to prove the case $X = \{x; |x(t)| \leq 1 \text{ a. e. } t \in E\}$. Let denote by A the set of all dyadic numbers in $[-1, 1]$ i. e. if $\lambda \in A$, then $\lambda = m/2^n$ for some integers $n(>0)$, m with $-2^n \leq m \leq 2^n$ and let $h_\lambda(t)$ ($\lambda \in A$) be Radon-Nikodym's derivative of $T(\lambda \mathcal{X}_e)$ i. e. $T(\lambda \mathcal{X}_e) = \int_E h_\lambda(t) d\mu$ where $\mathcal{X}_e$ is a characteristic function of a measurable set $e \subset E$. Now we take an arbitrary $\delta > 0$ and fix it for a while. For integers n and $m = -2^n + 1, \dots, 0, 1, \dots, 2^n$, we consider measurable sets:

$$e_m^n = \{t; |h_\lambda - h_\mu|(t) \geq \delta \text{ for some } \lambda, \mu \in A \text{ with } (m-1)/2^n \leq \lambda, \mu \leq m/2^n\} \quad (n=1, 2, \dots; m = -2^n + 1, \dots, 0, 1, \dots, 2^n)$$

and

$$b_n = \bigcup_{m=-2^n+1}^{2^n} e_m^n. \text{ Note that } e_m^n \text{ is written by } e_m^n = \bigcup_{\lambda, \mu} \{t; h_\lambda(t) - h_\mu(t) \geq \delta\}$$

where $\lambda, \mu (\in A)$ are countable. By definition of b_n , we have $b_1 \supset b_2 \supset \dots \supset b_n \supset \dots$, and let us denote their limit $b = \bigcap_{n=1}^{\infty} b_n$.

(i) Let $\mu(b) = 0$ for every $\delta > 0$. For each $\varepsilon > 0$, by (uac) , there exists $\varepsilon' > 0$ such that $|T(y\mathcal{X}_e)| < \varepsilon$ for all $y \in X$ and for $\mu(e) < \varepsilon'$. At the same time we see that there is n with $\mu(b_n) < \varepsilon'$ such that

$$|h_\lambda(t) - h_\mu(t)| < \varepsilon \text{ for } |\lambda - \mu| \leq 1/2^n \text{ with } \lambda, \mu \in A, t \in E \sim b_n.$$

This will be seen if we put $\delta = \varepsilon/2$. Let $\|x(t) - x'(t)\|_\infty < 1/2^{n+1}$. By (o) we can find $\lambda_i, \mu_i \in A$ (i : finite set) and mutually disjoint measurable sets e_i such that

$$\|x - \sum_i \lambda_i \mathcal{X}_{e_i}\|_\infty \leq 1/2^{n+2}, \quad \|x' - \sum_i \mu_i \mathcal{X}_{e_i}\|_\infty \leq 1/2^{n+2}$$

and

$$|T(x) - T(\sum_i \lambda_i \mathcal{X}_{e_i})| < \varepsilon, \quad |T(x') - T(\sum_i \mu_i \mathcal{X}_{e_i})| < \varepsilon.$$

We have $|\lambda_i - \mu_i| < 1/2^n$ for each i and

$$|T(\sum_i \lambda_i \mathcal{X}_{e_i}) - T(\sum_i \mu_i \mathcal{X}_{e_i})| \leq \sum_i \int_{e_i \sim b_n} |h_{\lambda_i} - h_{\mu_i}| d\mu$$

$$+ |T(\sum_i \lambda_i \mathcal{X}_{e_i} \mathcal{X}_{b_n})| + |T(\sum_i \mu_i \mathcal{X}_{e_i} \mathcal{X}_{b_n})| \leq \varepsilon \mu(E) + 2\varepsilon.$$

Hence,

$$\begin{aligned} |T(x) - T(x')| &\leq |T(x) - T(\sum_i \lambda_i \mathcal{X}_{e_i})| + |T(\sum_i \lambda_i \mathcal{X}_{e_i}) - T(\sum_i \mu_i \mathcal{X}_{e_i})| \\ &\quad + |T(x') - T(\sum_i \mu_i \mathcal{X}_{e_i})| \leq \varepsilon \mu(E) + 4\varepsilon. \end{aligned}$$

(uc_∞) follows from this.

(ii) Let $\mu(b) > 0$ for some $\delta > 0$. It will be shown that this case can not occur. For every n , we shall define a partition of b by induction, that is

$$b = b_{-2^{n+1}}^n \cup \cdots \cup b_{2^n}^n,$$

$b_m^n (m = -2^n + 1, \dots, 2^n)$ are mutually disjoint measurable sets and for every $p \geq n$, $b_m^n \subset e_m^{n,p}$ where $e_m^{n,p} = \{t; t \in e_{m'}^p \text{ for some } m' \text{ with } (m-1)/2^n < m'/2^p \leq m/2^n\}$. We take a note that the sequence $e_m^{n,p} (p = n, n+1, \dots)$

is monotone decreasing with respect to p and that $\bigcup_{m=-2^n+1}^{2^n} e_m^{n,p} = b_p$.

Let us define $b_m^1 (m = -1, 0, 1, 2)$, at first we put

$$a_m^1 = b \cap e_m^{1,1} \cap e_m^{1,2} \cap \cdots, \quad (m = -1, 0, 1, 2).$$

Then, we have $b = \bigcup_{m=-1}^2 a_m^1$, since $b \sim \bigcup_{m=-1}^2 a_m^1 = \bigcap_{m=-1}^2 \bigcup_{p=1}^\infty (b \sim e_m^{1,p}) \subset \bigcup_{p=1}^\infty \bigcap_{m=-1}^2 (b \sim e_m^{1,p}) \subset \bigcup_{p=1}^\infty (b \sim \bigcup_{m=-1}^2 e_m^{1,p}) = \phi$.

Hence we define, $b_{-1}^1 = a_{-1}^1$, $b_0^1 = a_0^1 \sim a_{-1}^1$, $b_1^1 = a_1^1 \sim a_0^1 \sim a_{-1}^1$, etc., then by definition $b_m^1 \subset a_m^1 \subset e_m^{1,p}$ for all $p \geq 1$. Assuming we have defined b_m^n , we shall define $b_s^{n+1}; s = -2^{n+1} + 1, \dots, 0, 1, \dots, 2^{n+1}$. It suffices to define b_{s-1}^{n+1} , b_s^{n+1} of the form $s = 2m$ with some $m \in \{-2^n + 1, \dots, 0, \dots, 2^n\}$. We set

$$b_s^{n+1} = b_m^n \cap e_s^{n+1, n+1} \cap e_s^{n+1, n+2} \cap \cdots,$$

$$b_{s-1}^{n+1} = b_m^n \sim b_s^{n+1}.$$

We must check $b_{s-1}^{n+1} \subset e_{s-1}^{n+1, p}$ for $p \geq n+1$. This will be done, since $t \in b_{s-1}^{n+1} \Rightarrow t \notin b_s^{n+1} \Rightarrow t \notin e_s^{n+1, p}$ for some $p \geq n+1 \Rightarrow t \notin e_s^{n+1, k}$ for all $k \geq p \Rightarrow t \in e_{s-1}^{n+1, k}$ for all $k \geq p \Rightarrow t \in e_{s-1}^{n+1, p}$ for all $p \geq n+1$, since $e_s^{n+1, k} \cup e_{s-1}^{n+1, k} = e_m^{n, k}$ by definition. For n , we define measurable functions $f_n, g_n \in X$ such that

$$f_n(t) = \sum_{m=-2^n+1}^{2^n} \frac{m-1}{2^n} \chi_{b_m^n}(t),$$

$$g_n(t) = \sum_{m=-2^n+1}^{2^n} \frac{m}{2^n} \chi_{b_m^n}(t).$$

We have by definition $f_n(t) \leq f_{n+1}(t) \leq g_{n+1}(t) \leq g_n(t)$, and $|f_n(t) - g_n(t)| \leq 1/2^n$. Hence, there exists $x \in X$ with $f_n \rightarrow x$, $g_n \rightarrow x$ (order convergence). For $m = -2^n + 1, \dots, 2^n$, we can decompose b_m^n into

$$b_m^n = c_{m,1}^n \cup c_{m,2}^n \cup \dots \quad (\text{mutually disjoint})$$

where $c_{m,i}^n \subset \{t; h_{\lambda_{m,i}^n}^n(t) - h_{\mu_{m,i}^n}^n(t) \geq \delta\}$ for every pair $\lambda_{m,i}^n, \mu_{m,i}^n \in A$ with $(m-1)/2^n \leq \lambda_{m,i}^n, \mu_{m,i}^n \leq m/2^n$, since $b_m^n \subset e_m^n = \bigcup_{\lambda, \mu} \{t; h_\lambda(t) - h_\mu(t) \geq \delta\}$ where $(m-1)/2^n \leq \lambda, \mu \leq m/2^n$ with $\lambda, \mu \in A$. If we define f'_n, g'_n as follows:

$$f'_n(t) = \sum_{m=-2^n+1}^{2^n} \sum_i \lambda_{m,i}^n \chi_{c_{m,i}^n}(t)$$

$$g'_n(t) = \sum_{m=-2^n+1}^{2^n} \sum_i \mu_{m,i}^n \chi_{c_{m,i}^n}(t),$$

then $f_n(t) \leq f'_n(t)$, $g'_n(t) \leq g_n(t)$ and

$$T(f'_n) - T(g'_n) \geq \delta \mu(b)$$

with $f'_n \rightarrow x$ and $g'_n \rightarrow x$ (order convergence). This contradicts to the condition (o).

Remark. In above Theorem, the condition (o) is replaced by the condition that T is continuous by measure convergence.

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