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UNITS IN COMMUTATIVE INTEGRAL GROUP RINGS*

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1. Introduction. Let us denote by $R(G)$ the group ring of a group G with coefficients from the ring R . Let $U(Z(G))$ be the group of units of the integral group ring $Z(G)$. It is well known (see [1], [3] and [4]) that if G is finite abelian

$$U(Z(G)) = \pm G \times F \quad \text{where}$$

F is a free group of rank $\frac{1}{2}((G:1) + 1 + n_2 - 2C)$ and n_2 is the number of elements of G of order 2 and C is the number of cyclic subgroups of G . In this note we compute $U(Z(G))$ for G arbitrary abelian and prove that

$$U(Z(G)) = \{\alpha g \mid g \in G, \alpha \in U(Z(H)) \text{ where } H \text{ is a finite subgroup of } G\}.$$

It is clear that when G is finite abelian any automorphism θ of $Z(G)$ is induced from a group automorphism i. e. $\theta(g) = \pm g$, $g \in G$ for $g \in G$. We compute the group of automorphisms of $Z(G)$ when G is finitely generated abelian.

2. Units of $Z(G)$.

Lemma 1. *If I is an integral domain and G a torsion free abelian group then the unit group of $I(G)$ is $U(I) \cdot G$.*

Proof. See [5].

Lemma 2. *Suppose R is a commutative ring with 0 and 1 as its only idempotents. Suppose $X = \langle x \rangle$ is an infinite cyclic group. Then the unit group of $R(X)$ is $U(R) \cdot X$ if R contains no nonzero nilpotent elements.*

Proof. Suppose $\gamma, \mu \in R(X)$ such that $\gamma\mu = 1$. We can take $\gamma = \sum_0^s a_i x^i$ and $\mu = \sum_{-t}^r b_j x^j$, $a_s \neq 0$. We first claim that $\mu = \sum_{-s}^0 b_j x^j$, $b_{-s} \neq 0$. Let P be a prime ideal of R which does not contain a_s . Then looking at $\gamma\mu = 1$ in $R/P(X)$ we conclude by Lemma 1 that $b_{-s} \neq 0$ and similarly if $b_{-t} \neq 0$ then $a_t \neq 0$. Thus $t = s$. By the same argument if $b_j \neq 0$ for some $j > 0$

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then $a_{-j} \neq 0$. We conclude that $\gamma = \sum_0^s a_i x^i$ and $\mu = \sum_0^s b_{-j} x^{-j}$.

Next we assert that $a_i a_j = 0$ and $a_i b_{-j} = 0$ for $i \neq j$. Suppose $a_i a_j \neq 0$ then choose a prime ideal P such that $a_i a_j \notin P$. Then $\bar{a}_i \neq 0$ and $\bar{a}_j \neq 0$ in R/P and $\bar{\gamma} \bar{\mu} = 1$ in the group ring $R/P(X)$. This is a contradiction to Lemma 1. Thus $a_i a_j = 0$ and similarly $a_i b_{-j} = 0$ and $b_{-i} b_{-j} = 0$ for $i \neq j$.

Now we have the situation

$$\left(\sum_0^s a_i x^i\right) \left(\sum_0^s b_{-j} x^{-j}\right) = 1$$

$$(*) \quad a_i b_{-j} = 0 = a_i a_j = b_{-i} b_{-j} \text{ for } i \neq j.$$

We can suppose $a_s b_{-s} \neq 0$. Then

$$a_0 b_0 + a_1 b_{-1} + \cdots + a_s b_{-s} = 1$$

Multiplying by a_s we have

$$a_s^2 b_{-s} = a_s \text{ and } (a_s b_{-s})^2 = a_s b_{-s}.$$

Thus $a_s b_{-s} = 1$ and from (*) $a_i = 0$ for $i \neq s$. Hence

$$\gamma = a_s x^s \text{ and } \mu = b_{-s} x^{-s}. \quad \text{Q. E. D.}$$

We remind the reader that a group G is said to be residually finite if $\bigcap_N N = 1$ where N runs over the normal subgroups of G with $(G:N)$ finite.

Lemma 3. Suppose G is a residually finite group and $Z(G)$ its integral group ring. Then

$$e \in Z(G), e^2 = e \Rightarrow e = 0 \text{ or } 1.$$

Proof. The corresponding result for G finite is well known (see [2]). Let $e = \sum_1^n \alpha_i g_i$, $\alpha_i \in Z$. Since G is residually finite one can choose a normal subgroup N of G such that $(G:N) < \infty$ and the cosets $g_i N$ $1 \leq i \leq n$ are all distinct. Consider the equation $e^2 = e$ in $Z(G/N)$. It follows then, say $g_1 \in N$, $\alpha_1 = 1$ or 0 and $\alpha_i = 0$ for $i > 1$. Hence $e = 0$ or 1 .

Lemma 4. Suppose G is an arbitrary abelian group. Then

$$e \in Z(G), e^2 = e \Rightarrow e = 0 \text{ or } 1.$$

Proof. This lemma follows from the last as G can be taken to be

finitely generated and hence residually finite.

Theorem 1. *Suppose G is an arbitrary abelian group. Then the unit group of $Z(G)$ is given by*

$$U(Z(G)) = \{ug \mid g \in G \text{ and } u \text{ is a unit of } Z(H) \text{ for some finite subgroup } H \text{ of } G\}.$$

Proof. Let $\bar{r} \in U(Z(G))$. we can suppose G is finitely generated and

$$G = T \times \langle x_1 \rangle \times \cdots \times \langle x_s \rangle, \quad |T| < \infty.$$

We use induction on s . Since $Z(T \times \langle x_1 \rangle \times \cdots \times \langle x_{s-1} \rangle)$ has no nontrivial idempotent or nilpotent elements, we can apply Lemma 2 to conclude that $\bar{r} = \bar{r}_1 x_s^i$ where $\bar{r}_1$ is a unit of $Z(G_1)$, $G_1 = T \times \langle x_1 \rangle \times \cdots \times \langle x_{s-1} \rangle$. Now by induction $\bar{r}_1 = ug_1$ where $g_1 \in G$ and u is a unit of $Z(H)$, $|H| < \infty$. Thus $\bar{r} = ug_1 x_s^i = ug$.

Corollary. *If G is a finitely generated abelian group, say, $G = T \times F$ where T is finite and F is free. Then the unit group of $Z(G)$ is given by $U_1 \times F$ where U_1 is the unit group of $Z(T)$.*

3. Automorphisms of $Z(G)$. Since for G finite abelian, the only units of finite order in $Z(G)$ are $\pm g$ for $g \in G$ (see [4]) it follows that all automorphisms of $Z(G)$ are induced from (group) automorphisms of G . We have also proved (see [5]) that all automorphisms of $Z(G)$ when G is torsion or torsion free abelian, are induced from automorphisms of G . But this need not be the case when G is mixed. For example, let $G = \langle g \rangle \times \langle x \rangle$, $g^5 = 1$ and $o(x) = \infty$. Then U the unit group of $Z(G)$ is given by $\pm G \times \langle u \rangle$, $o(u) = \infty$. This is because the free part in the unit group of $Z(\langle g \rangle)$ has rank one (see [1] and [3]). Define $\theta : Z(G) \rightarrow Z(G)$ by

$$\theta(\sum a_{ij} g^i x^j) = \sum a_{ij} g^i u^j x^j.$$

It is easy to see that θ is a homomorphism. This is a one to one map as $0 = \sum a_{ij} g^i u^j x^j \Rightarrow \sum_i a_{ij} g^i u^j = 0 \Rightarrow \sum a_{ij} g^i = 0 \Rightarrow a_{ij} = 0$.

Let G be a finitely generated abelian group. Then $G = T \times F$, where T is finite and F is free. The unit group of $Z(G)$ is given by

$$\begin{aligned} U(Z(G)) &= T \times U_1 \times F, \text{ where} \\ U(Z(T)) &= T \times U_1. \end{aligned}$$

Suppose

A = Automorphism group of $Z(G)$

A_1 = Group of those automorphisms of $Z(G)$ which are induced by automorphisms of G (i. e. $\theta(g) = \pm g_1$; $g, g_1 \in G$)

A_2 = Group of those automorphisms of $T \times U_1 \times F$ which keep $T \times U_1$ element wise fixed.

Theorem 2. $A = A_1 \times A_2$.

Proof. Given any element θ of A , define

$$\lambda(g) = \begin{cases} \theta(g) & \text{for } g \in T \\ g & \text{for } g \in F. \end{cases}$$

Extend λ to an automorphism of $Z(G)$. Then $\lambda \in A_1$ and $\lambda^{-1}\theta$ keeps $T \times U_1$ element wise fixed and can be considered as an element of A_2 .

Conversely, given $\lambda \in A_1$, $\mu \in A_2$, define

$$\theta(\sum_i u_i f_i) = \lambda(\sum_i u_i \mu(f_i)),$$

where $f_i \in F$ are distinct and $u_i \in Z(T)$. θ is clearly a homomorphism.

Suppose $\theta(\sum_i u_i f_i) = 0$. Then

$$0 = \theta(\sum_i u_i f_i) = \lambda(\sum_i u_i \mu(f_i)).$$

Therefore, $0 = \sum u_i \mu(f_i) = \sum u_i v_i f'_i$, $v_i \in T \times U_1$, $f'_i \in F$.

Therefore $u_i = 0$ due to distinctness of f'_i . Hence, $\sum u_i f_i = 0$.

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