

Mathematical Journal of Okayama University

Volume 14, Issue 2

1969

Article 10

DECEMBER 1970

On separable extensions of domains

Takasi Nagahara*

*Okayama University

Copyright ©1969 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

ON SEPARABLE EXTENSIONS OF DOMAINS

TAKASI NAGAHARA

Throughout this paper we shall understand by a domain a commutative ring with identity element and without zero divisors $\neq 0$, and all rings will be assumed commutative with identity element. Let B be a domain. If the quotient field of B contains no finitely generated separable B -subalgebras containing B but not coinciding with B then we say that B is a *separably closed domain*. Clearly any integrally closed domain is separably closed. In Th. 1, we see that B is a separably closed domain if and only if every finitely generated, separable B -algebra containing B but not containing proper idempotents (i. e., not containing idempotents except 0 and 1) is a domain which is projective over B . In virtue of the result, we prove some structure theorems of separable algebras over a separably closed domain, containing a Janusz's result [6, Th. 4. 3] which is a structure theorem of finitely generated separable algebras over a Noetherian integrally closed domain.

In what follows, we shall summarize the notations and definitions which will be used in the subsequent study. For a ring S and for a subring R , we denote by $\mathfrak{A}(S/R)$ the group of (ring) automorphisms in S which leave the elements of R fixed, and moreover, for a set Φ of (ring) automorphisms in S , we denote by $\Phi|_R$ the restriction of Φ to R and denote by $J(\Phi)$ (resp. by $J(\Phi|_R)$) the fixring of Φ in S (resp. $J(\Phi) \cap R$).

As in [6] and [8], an R -algebra S will be called respectively

f. g. separable, if S is finitely generated and separable over R ;

locally f. g. separable, if every finite set of elements in S is contained in an f. g. separable R -subalgebra of S ;

strongly separable, if S is finitely generated, projective, and separable over R ;

locally strongly separable, if every finite set of elements in S is contained in a strongly separable R -subalgebra of S .

The property of being f. g. separable (resp. strongly separable) is transitive (see the proof of [2, Th. 2. 3]). From this, it follows easily that the property of being locally f. g. separable (resp. locally strongly separable) is transitive.

Now our study starts with the following

Proposition 1. *Let A be a ring without proper idempotents, and $R \subset S \subset T$ subrings of A . If T is f. g. separable over R and $J(\mathfrak{S}(A/S))=S$ then*

- (1) T is strongly separable over S and $J(\mathfrak{S}(A/T))=T$,
- (2) S is f. g. separable over R .

Proof. (1) is contained [8, Th. 1], and the argument used in the proof of [2, Th. 2. 3] enables us to obtain the remaining.

Corollary 1. *Let A be a ring without proper idempotents, and $R \subset S \subset T$ subrings of A . If T is locally f. g. separable over R and $J(\mathfrak{S}(A/S))=S$ then*

- (1) T is locally strongly separable over S , and
- (2) S is locally f. g. separable over R .

Proof. Obviously T is locally f. g. separable over S . Then by Prop. 1, T is locally strongly separable over S . Let F be a finite subset of S , and T' a subring of T containing $R[F]$ which is f. g. separable over R . Then, by making use of the same method as in the proof of [8, Th. 1], we can prove that $\mathfrak{S}(A/S)|T'$ is a finite set. Hence the subring A' of A generated by $\bigcup_{\sigma \in \mathfrak{S}(A/S)} \sigma(T')$ is an f. g. separable R -algebra, and $\mathfrak{S}(A/S)|A'$ is a group of automorphisms in A' . By Prop. 1, $J(\mathfrak{S}(A/S)|A')$ is an f. g. separable R -algebra which is a subring of S containing F . Thus S is locally f. g. separable over R .

As in [6], if R is a ring without proper idempotents then there exists a locally strongly separable R -algebra, $\mathcal{Q}$, such that $\mathcal{Q}$ has no proper idempotents and if Γ is a strongly separable $\mathcal{Q}$ -algebra with no proper idempotents then $\Gamma=\mathcal{Q}$ ([6, Prop. 1. 4]). Such an R -algebra will be called a *separable closure* of R . The separable closure $\mathcal{Q}$ of R is unique up to isomorphisms, and every locally strongly separable R -algebra without proper idempotents can be imbedded in $\mathcal{Q}$ ([6, Prop. 1. 7]). Moreover there holds $J(\mathfrak{S}(\mathcal{Q}/R))=R$ ([6, Prop. 1. 9]). Combining these results with Coro. 1 and [8, Th. 3], we can easily obtain the following

Proposition 2. *Let A be a ring without proper idempotents which contains subrings $R \subset S$, and let A be locally strongly separable over R . Then, A is locally strongly separable over S if and only if S is locally f. g. separable over R . If the conditions hold then S is locally*

strongly separable over R .

Similarly we have the following

Corollary 2. *Let A be a ring without proper idempotents which contains subrings $R \subset S$, and let A be locally f. g. separable over R , and $J(\mathfrak{F}(A/R)) = R$. Then A is locally strongly separable over R , and the following conditions are equivalent.*

- (a) $J(\mathfrak{F}(A/S)) = S$.
- (b) A is locally strongly separable over S .
- (c) S is locally f. g. separable over R .
- (d) S is locally strongly separable over R .

Throughout the rest of this paper B will denote a domain, and we shall use the following conventions :

$\mathcal{Q}$: the separable closure of B .

Q : the quotient field of B .

$\bar{Q}$: the algebraic closure of Q .

$\bar{B}$: the integral closure of B in $\bar{Q}$ (i. e., the totality of elements of $\bar{Q}$ that are integral over B .)

Now we shall prove the following theorem which contains the results of [6, Lemma 4. 1 and Coro. 4. 2].

Theorem 1. *The following conditions are equivalent.*

- (a) B is a separably closed domain.
- (b) Every domain containing B which is f. g. separable over B is projective over B .
- (c) Every ring containing B which has no proper idempotents and is f. g. separable over B is a domain and is projective over B .

Proof. We shall give a cyclic proof of this theorem in order : (a) implies (b), (b) implies (c), (c) implies (a). Assume (a). Let R be a domain containing B which is f. g. separable over B . Then $Q[R]$ is an f. g. separable field extension of Q . Hence $Q[R]$ is imbedded in a field K which is Galois over Q . Let S be the subring of K generated by $\bigcup_{\sigma \in \mathfrak{F}(K/Q)} \sigma(R)$. Then S is an f. g. separable extension of B and $\mathfrak{F}(K/Q)|S$ is a group of automorphisms of S . By Prop. 1, S is a strongly separable extension of $J(\mathfrak{F}(K/Q)|S)$ and $J(\mathfrak{F}(K/Q)|S)$ is an f. g. separable extension of B which is a subring of Q . Since B is a separably closed domain, it follows that $J(\mathfrak{F}(K/Q)|S) = B$; hence R is projective over B . Thus we have (b). Assume (b). Let R be an f. g.

separable ring extension of B without proper idempotents. Then $Q \otimes_B R$ is a direct sum of separable field extensions of Q ([9, Th. 1]). For the identity element e of one of the fields in this decomposition, R is B -algebra homomorphic to an f. g. separable domain $(1 \otimes R)e$ containing B . By (b), $(1 \otimes R)e$ is projective over B . Hence by [6, Lemma 1.6], we have a B -algebra isomorphism $R \cong (1 \otimes R)e$. Thus we obtain (c). Assume (c). Let R be a subring of Q containing B which is f. g. separable over B . Then R is projective over B . Hence by [6, Th. 1.1], R is imbedded in a Galois extension S of B without proper idempotents. Then S is a domain too. Since Q is the quotient field of B and R is a subring of Q containing B , we have $R \subset J(\mathfrak{S}(S/B)) = B$, and hence $R = B$. Therefore B is a separably closed domain. This completes the proof.

Remark 1. As in an example of G. J. Janusz ([6, P. 473]), we consider a ring $R = Z_{(2)} + Z_{(2)}\sqrt{5}$ where $Z_{(2)}$ is the localization at (2) in the ring of rational integers. Then $R[X]/(X^2 + X - 1)$ is a strongly separable R -algebra which has no proper idempotents and is not a domain. Hence by Th. 1, R is not separably closed.

Theorem 2. B is a separably closed domain if and only if $\mathcal{Q}$ is a domain and is separably closed.

Proof. Let B be a separably closed domain. Then $\mathcal{Q}$ is a domain by Th. 1 ((a) $\Rightarrow$ (c)). Let Γ be a domain containing which is f. g. separable over $\mathcal{Q}$. Then Γ is locally f. g. separable over B . By Th. 1 ((a) $\Rightarrow$ (b)), Γ is locally strongly separable over B . Hence $\Gamma \subset \mathcal{Q}$, and this implies $\Gamma = \mathcal{Q}$. Thus $\mathcal{Q}$ is separably closed by Th. 1 ((b) $\Rightarrow$ (a)). Assume that $\mathcal{Q}$ is a domain and is separably closed. Let R be a domain containing B which is f. g. separable over B . Then $\mathcal{Q}[R] (\subset \bar{B})$ is f. g. separable over $\mathcal{Q}$. By Th. 1 ((a) $\Rightarrow$ (b)), $\mathcal{Q}[R]$ is projective over $\mathcal{Q}$. This gives $\mathcal{Q}[R] = \mathcal{Q}$, and so, R is projective over B by Prop. 2. Hence B is separably closed by Th. 1 ((b) $\Rightarrow$ (a)).

Corollary 3. Let B be a separably closed domain. Then every locally f. g. separable extension of B without proper idempotents is a separably closed domain which is a subring of $\mathcal{Q}$. In particular, every f. g. separable extension of B without proper idempotents is a separably closed domain which is projective over B .

Proof. Let R be a locally f. g. separable extension of B without proper idempotents. Then by Th. 1, R is a domain and is locally strongly separable over B . Hence R is a subring of $\mathcal{Q}$, and by Prop. 2, $\mathcal{Q}$ is the separable closure of R . Therefore by Th. 2, R is separably closed.

Corollary 4. *Let B be a separably closed domain, and R an intermediate ring of $\mathcal{Q}/B$. Then, R is a separably closed domain if and only if R is a locally f. g. separable B -algebra.*

Proof. Clearly $\mathcal{Q}$ is locally f. g. separable over R . If R is a separably closed domain then, by Th. 1, $\mathcal{Q}$ is locally strongly separable over R ; then, by Prop. 2, R is locally f. g. separable over B . The converse part is a direct consequence of Coro. 3.

Combining Coro. 4 with [8, Th. 3], we have the following

Corollary 5. *Let B be a separably closed domain. Then there exists a 1-1 dual correspondence between separably closed, intermediate rings of $\mathcal{Q}/B$ and subgroups of $\mathfrak{S}(\mathcal{Q}/B)$ which are closed in the finite topology, in the usual sense of Galois theory.*

Remark 2. Let B be a domain. Let $R \supset B$ and $S \supset B$ be subrings of $\bar{Q}$ such that R is a separably closed domain, and S is a locally f. g. separable B -algebra satisfying $J(\mathfrak{S}(S/B)) = B$. Then $J(\mathfrak{S}(R[S]/R)) = R$ and $J(\mathfrak{S}(S/R \cap S)) = R \cap S$. Hence by Coro. 2, $S/R \cap S$ and $R \cap S/B$ are locally strongly separable extensions. Moreover, we have $R[S] \cong R \otimes_{R \cap S} S$ and $\mathfrak{S}(R[S]/R) \cong \mathfrak{S}(S/R \cap S)$. Next, let $\mathcal{Q}'$ be the subring of $\bar{B}$ generated by all f. g. separable B -subalgebras, and set $B' = J(\mathfrak{S}(\mathcal{Q}'/B))$. Then $\mathcal{Q}'$ is locally f. g. separable over B and is the separable closure of B' ; and B' is a subring of the quotient field of B which is separably closed and is locally f. g. separable over B .

Remark 3. Let B be a separably closed domain, and A an f. g. separable B -algebra which is B -torsion free. Then A is a direct sum of separably closed domains each of which is strongly separable over B , and A can be imbedded in a Galois extension of B (cf. [10]).

Moreover, by the same method as in the proof of [6, Th. 4.3], we can prove the following

Theorem 3. *Let B be a separably closed domain. If A is an f. g.*

separable B -algebra then there exists an idempotent e of A such that $A = Ae \oplus t(A)$ where $t(A)$ is the B -torsion submodule of A and Ae is a direct sum of separably closed domains each of which is strongly separable over B .

In the rest of this note, we shall make some remarks on separable algebras over integrally closed domains. It is clear that every integrally closed domain is a separably closed domain. We now prove

Theorem 4. *B is an integrally closed domain if and only if $\mathcal{Q}$ is a domain and is integrally closed.*

Proof. Let B be an integrally closed domain. Then $\mathcal{Q}$ is a domain by Th. 2. Let a/b be an element of the quotient field of $\mathcal{Q}$ which is integral over $\mathcal{Q}$ where $a, b \in \mathcal{Q}$. Then $a/b \in Q[a, b] \cap \bar{B} \subset Q[\mathcal{Q}] \cap \bar{B}$. Since $J(\mathfrak{S}(Q[\mathcal{Q}] \cap \bar{B}/B)) = Q \cap \bar{B} = B$, it follows from Prop. 1 that $a/b \in J(\mathfrak{S}(Q[\mathcal{Q}] \cap \bar{B}/B[a, b])) \subset \mathcal{Q}$. Hence $\mathcal{Q}$ is integrally closed. Conversely, if $\mathcal{Q}$ is a domain and is integrally closed then we have $\mathcal{Q} = Q[\mathcal{Q}] \cap \bar{B}$ and $B = J(\mathfrak{S}(Q[\mathcal{Q}]/Q) | \mathcal{Q}) = J(\mathfrak{S}(Q[\mathcal{Q}]/Q) | Q[\mathcal{Q}] \cap \bar{B}) = Q \cap \bar{B}$; hence B is integrally closed.

The following corollary contains the results of [6, Coro. 4.2] and [7, Coro. 5.2].

Corollary 6. *Let B be an integrally closed domain. Then every locally f. g. separable extension of B without proper idempotents is an integrally closed domain which is a subring of $\mathcal{Q}$. In particular, every f. g. separable extension of B without proper idempotents is an integrally closed domain which is projective over B .*

Proof. If R is a locally f. g. separable extension of B without proper idempotents then $\mathcal{Q}$ is the separable closure of R . Hence by Th. 4, R is an integrally closed domain.

Remark 4. Let B be an integrally closed domain, and R an intermediate ring of $\mathcal{Q}/B$. Then, by Coro. 4 and Coro. 6, R is integrally closed if and only if R is locally f. g. separable over B ; this is equivalent to that R is separably closed. Hence by Coro. 5, there exists a 1—1 dual correspondence between integrally closed, intermediate rings of $\mathcal{Q}/B$ and subgroups of $\mathfrak{S}(\mathcal{Q}/B)$ which are closed in the finite topology, in the usual sense of Galois theory. However, in general, the subring Γ of $\bar{B}$ consisting of elements which are separable over Q is an integrally closed domain containing $\mathcal{Q}$, and there exists a 1—1 dual correspondence between

integrally closed, intermediate rings Γ/B and subgroups of $\mathfrak{S}(\Gamma/B)$ which are closed in the finite topology, in the usual sense of Galois theory.

Remark 5. Let B be an integrally closed domain. If A is an f. g. separable B -algebra then there exists an idempotent e of A such that $A = Ae \oplus t(A)$ where $t(A)$ is the B -torsion submodule of A and Ae is a direct sum of integrally closed domains each of which is strongly separable over B . This contains the result of [6, Th. 4. 3].

REFERENCES

- [1] M. AUSLANDER and D. BUCHSBAUM: On the ramification theory in Noetherian rings, Amer. J. Math. 81 (1959), 749—765.
- [2] M. AUSLANDER and O. GOLDMAN: The Brauer group of a commutative rings, Trans. Amer. Math. Soc. 97 (1960), 367—409.
- [3] N. BOURBAKI: Algèbre commutative, Chapitre I-II, Actualités Sci. Ind. No. 1290, Hermann, Paris, 1962.
- [4] S. U. CHASE, D. K. HARRISON and A. ROSENBERG: Galois theory and cohomology of commutative rings, Mem. Amer. Math. Soc. No. 52 (1965).
- [5] M. HONGAN and T. NAGAHARA: A note on separable extensions of commutative rings, Math. J. of Okayama Univ., 14 (1969), 13—15.
- [6] G. J. JANUSZ: Separable algebras over commutative rings, Trans. Amer. Math. Soc. 122 (1966), 461—479.
- [7] K. KISHIMOTO: On abelian extensions of rings I, Math. J. of Okayama Univ., 14 (1970), 159—174.
- [8] T. NAGAHARA: A note on Galois theory of commutative rings, Proc. Amer. Math. Soc. 18 (1967), 334—340.
- [9] A. ROSENBERG and D. ZELINSKY: Cohomology of infinite algebras, Trans. Amer. Math. Soc. 82 (1956), 85—98.
- [10] O. E. VILLAMAYOR and D. ZELINSKY: Galois theory for rings with finitely many idempotents, Nagoya Math. J. 27 (1966), 721—731.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

(Received June 17, 1970)