Mathematical Journal of Okayama University

Volume 14, *Issue* 2 1969 *Article* 4

DECEMBER 1970

Discrete analytic derivative equations of the second order

Shih-Tong Tu*

*Okayama University

DISCRETE ANALYTIC DERIVATIVE EQUATIONS OF THE SECOND ORDER

SHIH-TONG TU

1. Introduction. This paper is concerned to discuss about the general solution of the discrete analytic derivative equations $\frac{\partial^2 F}{\partial z^2} - aK*F(z) = b(z)$ with the initial conditions $\frac{\partial F(0)}{\partial z} = C_2$ and $F(0) = C_1$. Throughout this paper, we need a few definition and some notations, such as "discrete analytic function", "region", "derivative", "line integral", "convolution", "double dot integral", "$A(R)$", "$*$"; and "L"; these are mentioned in [1].

In [2], Duffin and Duris has discussed about the general solution of discrete derivative equation of the first order with constant coefficient. If $a^4 \neq 16$, then the general solution of $\frac{\partial F(z)}{\partial z} - aF(z) = b(z)$ with $F(0) = C$, where $b(z) \in A(R)$ is $F(z) = C\,e(z, a) + \int_0^z e(z-t, a) : b(t)\delta t$ where C is an arbitrary constant, and $e(z, a) = \left(\frac{2+a}{2-a}\right)^x \left(\frac{2+ai}{2-ai}\right)^y$ is known as the discrete exponential function which is introduced ay Ferrand [3].

Afterwords, in [1], present author has developed a theory to general case, if $K(z) \in A(R)$, R contains the origin and $ah^2[K(0)+K(h)] \neq 8$ for $h = \pm 1$ or $\pm i$, then there exists a unique analytic function $F(z)$ in R, such that $\frac{\partial F(z)}{\partial z} - aK(z)*F(z)$ with $F(0) = C$, where $b(z) \in A(R)$. For the type of second order equations $\frac{\partial^2 F(z)}{\partial z^2} - aK*F(z) = b(z)$ with $\frac{\partial F(0)}{\partial z} = C_2$ and $F(0) = C_1$, we have analogous properties to the first order.

2. Discrete derivative equations of the type $\frac{\partial^2 F}{\partial z^2} - aK(z)*F(z) = 0$. In [4], Hayabara has shown the following theorem in operational sense.

Theorem 1. 1. $f \in A(R)$

$$\Rightarrow n! \int_0^z \int_0^{t_1} \cdots \int_0^{t_n} f(t_{n+1}) \delta t_{n+1} \cdots \delta t_1 = \int_0^z (z-t)^{(n)} : f(t) \delta t$$

where $z^{(n+1)} = (n+1) \int_0^z t^{(n)} \delta t$, $z^{(0)} = 1$.

Mathematical Journal of Okayama University, Vol. 14 [1969], Iss. 2, Art. 4

In [5], Duffin and Duris have solved a discrete Volterra integral equations.

Theorem 2. 1. *$f(z)$, $K(z)\in A(R)$ where R contains the origin*
2. $ah[G(0)+G(h)]\neq 4$ *for* $h=\pm 1$ *or* $\pm i$

$\Rightarrow$ *there exists a unique function* $F(z)\in A(R)$

such that $F(z)=f(z)+a\int_0^z G(z-t):F(t)\delta t.$

And the solution can be calculated by stepping formula (1).

$$(1)\quad F(z+h)=\frac{1}{4-ah[G(0)+G(h)]}\Big\{4f(z+h)+ah[G(0)+G(h)]F(z)+4a\int_0^z G(z+h-t):F(t)\delta t\Big\}$$

with $F(0)=f(0)$.

Theorem 3. 1. *$K(z)\in A(R)$ where R contains the origin*
2. $16-ah^3[K(0)+K(h)]=0$

$\Rightarrow$ (2) $\dfrac{\partial^2 F}{\partial z^2}-aK*F(z)=0$ *with* $\dfrac{\partial F(0)}{\partial z}=C_2$ *and* $F(0)=C_1$

has no solution for $z=h$ *if* $C_2=0$, $C_1\neq 0$ *or* $C_2\neq 0$, $C_1=0$.

Proof. Suppose, there exists a solution of (2) for $z=h$, with $C_2=0$, $C_1\neq 0$ or $C_2\neq 0$, $C_1=0$. Let $M(z)=K*F(z)$, from (2) we have
$a\int_0^h M(t)\delta t=\int_0^h\frac{\partial^2 F}{\partial z^2}\delta z=\frac{\partial F(h)}{\partial z}-C_2$ *i. e.* $\frac{\partial F(h)}{\partial z}=\frac{ah}{2}M(h)+C_2.$

By the definition of the derivative (see [1]), we have $\frac{\partial F(h)}{\partial z}=\frac{2}{h}[F(h)-C_1]-C_2.$

Therefore,
$$\frac{2}{h}[F(h)-C_1]-C_2=\frac{ah}{2}\int_0^h K(h-t):F(t)\delta t+C_2$$
$$=\frac{ah^2}{8}[K(0)+K(h)][F(h)+C_1]+C_2$$

i. e. $\{16-ah^3[K(0)+K(h)]\}F(h)=\{16+ah^3[K(0)+K(h)]\}C_1+16hC_2$. Hence, $16-ah^3[K(0)+K(h)]=0$ and if for $C_2=0$, $C_1\neq 0$ it contradicts to assumption. For $C_2\neq 0$, $C_1=0$ it is also a contradiction. Thus, this proves the theorem.

Theorem 4. *Let $K(z)$ be discrete analytic in R containing the origin. And if* $16-ah^3[K(0)+K(h)]\neq 0$ *for h equals to one of the values* ± 1 *or* $\pm i$. *Then there exists a unique function $F(z)$ discrete analytic in R such that*

$\frac{\partial^2 F}{\partial z^2} - aK*F(z)=0$ *with* $F(0)=C_1$ *and* $\frac{\partial F(0)}{\partial z}=C_2$. *And the solution of* (2) *can be calculated by the following stepping formula*:

$$\text{(3)}\quad F(z+h)=\frac{1}{16-ah^3[K(0)+K(h)]}\Big\{16[C_2(z+h)+C_1]+ah^3[K(0)+K(h)]F(z)+16a\int_0^z G(z+h-t):F(t)\delta t\Big\}$$

where $G(z)=z*K(z)$.

Proof. Suppose, (2) has a solution in R and let $K(z)*F(z)=M(z)$. Then we obtain $\frac{\partial F(z)}{\partial z}=a\int_0^z M(t)\delta t+C_2$,

and $F(z)=a\int_0^z\int_0^{t_1} M(t)\delta t\delta t_1+C_1+C_2 z$.

By using Theorem 1, it becomes discrete Volterra integral equation, such as $F(z)=a\int_0^z (z-t):M(t)\delta t+C_2 z+C_1=C_2 z+C_1+aG*F(z)\cdots\cdots(4)$.

For a fixed chain $(z_0,\cdots,z_m)$ from 0 to z in R, we have

$$L\ F(z)=L\ (C_2 z+C_1)+a\ L\int_0^z G(z-t):F(t)\delta t$$

Since, $L\ (C_2 z+C_1)=0$ (assume $a\neq 0$)
we can obtain the following four expressions (see [5] pp. 210—211)

$\{4-ai[G(0)+G(i)]\}L\ F(z)=0$

or $\{4+a[G(0)+G(-1)]\}L\ F(z)=0$

or $\{4+ai[G(0)+G(-i)]\}L\ F(z)=0$

or $\{4-a[G(0)+G(1)]\}L\ F(z)=0$.

But, $G(0)=0$ and $G(\mathrm{h})=\int_0^h (h-t):K(t)\delta t=\frac{h^2}{4}[K(0)+\mathrm{K}(h)]$

Hence, above four expressions become the following forms respectively.

$\{16+ai[K(0)+K(i)]\}L\ F(z)=0$

or $\{16+a[K(0)+K(-1)]\}L\ F(z)=0$

or $\{16-ai[K(0)+K(-i)]\}L\ F(z)=0$

or $\{16-a[K(0)+K(1)]\}L\ F(z)=0$.

Thus. if $16-ah^3[K(0)+K(h)]\neq 0$ for h equal to one of the values ± 1 or $\pm i$, then $LF(z)=0$. This proves that if (2) has a solution in R, then this solution is discrete analytic in R. By theorem 2, there exists a unique solution $F(z)$ of (4) discrete analytic in R. And $F(z)$ is uniquely determined by the following stepping formula.

$$F(z+h)=\frac{1}{4-ah[G(0)+G(h)]}\Big\{4[C_2(z+h)+C_1]+ah[G(0)+G(h)]F(z)$$
$$+4a\int_0^z G(z+\mathrm{h}-t):F(t)\delta t.\Big\}.$$

On the other hand, we can rewrite $F(z+h)$ into the following form.

$$(3)\quad F(z+h)=\frac{1}{16-ah^3[K(0)+K(h)]}\Big\{16[C_2(z+h)+C_1]+a\mathrm{h}^3[K(0)$$
$$+K(h)]F(z)+16a\int_0^z G(z+h-t):F(t)\delta t\Big\}$$

where $G(z)=z*K(z)$.

(3) is the required stepping formula for finding the unique solution $F(z)$ of (2). Now it remains to prove that the function $F(z)$ which is obtained uniquely from (3), is exactly a solution of (2). Throughout the following proof, we use some notations. $\overline{K}(n)=K(n)+K(n-1)$ where n is a positive integer. And let $B=16-a\overline{K}(1)$, from (3) we obtain $BF(1)=16(C_1+C_2)+aC_1\overline{K}(1)$. Substituting $F(1)$ into (2), we easily see that (2) has a solution for $z=1$. Before we prove that (2) has a solution for $z=2, 3, 4, \cdots\cdots$, we need the following lemmas. The first is easy from (3).

Lemma 1.

$$(5)\qquad \overline{G}(n)=\sum_{i=1}^{n-1} i\overline{K}(n-i)+\frac{1}{4}\overline{K}(n)$$

$$(6)\qquad BF(n+1)=16[C_2(n+1)+C_1]+a\overline{K}(1)F(n)$$
$$+4a\sum_{j=2}^{n+1}\sum_{i=1}^{j-1} i\overline{K}(j-i)\overline{F}(n-j+2)+a\sum_{j=2}^{n+1}\overline{K}(j)\overline{F}(n-j+2)$$

Lemma 2. $p\geqq 4$

$$(7)\qquad E\equiv F(p-2)[12\overline{K}(1)+7\overline{K}(2)+2\overline{K}(3)]+[8\overline{G}(3)-12\overline{K}(1)-5\overline{K}(2)$$
$$-\overline{K}(3)]F(p-3)-4\overline{G}(3)F(p-4)+8[\overline{G}(p)\overline{F}(1)+\cdots+\overline{G}(4)\overline{F}(p-3)]$$
$$-4[\overline{G}(p+1)\overline{F}(1(+\cdots+\overline{G}(4)\overline{F}(p-2)]+\{\overline{\overline{\overline{K}}}(p+1)\overline{F}(1)+\cdots$$
$$+\overline{\overline{\overline{K}}}(4)\overline{F}(p-2)\}-4[\overline{G}(p-1)\overline{F}(1)+\cdots+\overline{G}(4)\overline{F}(p-4)]\equiv 0.$$

Proof. Rearranging the left-hand side into the polynomial with respect to $F(i)$, where $i=0, 1, \cdots, p-2$. We see easily that every coefficient of the term $F(i)$ equals zero. Thus, this lemma is proved.

Lemma 3. *For* $n\geqq 2$, *we have*

$$(8)\qquad 4\{F(n)-3F(n-1)+4F(n-2)-4F(n-3)+\cdots+(-1)^{n+1}4F(1)$$

$$+(-1)^n 2F(0)+(-1)^n C_2\}$$
$$=aK*F(n)+aK*F(n-1), \quad where \ \frac{\partial F(0)}{\partial z}=C_2$$

Proof. It holds for $n=2$. Suppose, (8) is true for $n=p$.

(9) $$4\{F(p)-3F(p-1)+4F(p-2)-4F(p-3)+\cdots+(-1)^{p+1}4F(1)+(-1)^p 2F(0)+(-1)^p C_2\}=aK*F(p)+aK*F(p-1).$$

We want to claim that

(10) $$4\{F(p+1)-3F(p)+\cdots+(-1)^{p+2}4F(1)+(-1)^{p+1}2F(0)+(-1)^{p+1}C_2\}=aK*F(p+1)+aK*F(p).$$

From (9) and (10), we get

(11) $$4\Big\{F(p+1)-2F(p)+F(p-1)\Big\}=a\int_0^{p+1}K(p+1-t):F(t)\partial t+2a\int_0^p K(p-t):F(t)\partial t+a\int_0^{p-1}K(p-1-t):F(t)\partial t.$$

Therefore, for proving (10), it is sufficient to show (11).

Since $\int_0^p K(p-t):F(t)\partial t=\frac{1}{4}\sum_{r=1}^{p}\overline{K}(p-z_{r-1})\overline{F}(z_r)$, where $z_r=r$,

we have

$$W\equiv \text{Right-hand side of (11)}=\frac{a}{4}\Big\{\overline{\overline{\overline{K}}}(p+1)\overline{F}(1)+\cdots+\overline{\overline{\overline{K}}}(4)\overline{F}(p-2)\Big\}+\frac{a}{4}\overline{\overline{\overline{K}}}(3)\overline{F}(p-1)+\frac{a}{4}[\overline{K}(2)+2\overline{K}(1)]\overline{F}(p)+\frac{a}{4}\overline{K}(1)\overline{F}(p+1).$$

Let $V\equiv$ Left-hand side of (11).

Then, rewriting (11) into the form

$$4(W-V)=-BF(p+1)+16[2F(p)-F(p-1)]+a\Big\{\sum_{j=1}^{p-2}\overline{\overline{\overline{K}}}(j+3)\overline{F}(p-j-1)+F(p-2)\overline{\overline{\overline{K}}}(3)+F(p-1)[\overline{K}(3)+3\overline{K}(2)+3\overline{K}(1)]+F(p)[\overline{K}(2)+3\overline{K}(1)]\Big\},$$

from (6), we get

$$4(W-V)=2BF(p)-16[C_2(p+1)+C_1]+F(p-1)\{-9a\overline{K}(1)-2a\overline{K}(2)-16\}+aF(p-2)\{-7\overline{K}(1)-2\overline{K}(2)\}-4a\{\overline{G}(p+1)\overline{F}(1)+\cdots$$

$$+\overline{G}(4)\overline{F}(p-2)\}+a[\overline{\overline{\overline{K}}}(p+1)\overline{F}(1)+\cdots+\overline{\overline{\overline{K}}}(4)\overline{F}(p-2)].$$

Again, from (6), we have

$$\begin{aligned}4(W-V)=&-BF(p-1)+16(C_2(p-1)+C_1)+F(p-2)\{17a\overline{K}(1)+8a\overline{K}(2)\\&+2a\overline{K}(3)\}+8a\{\overline{G}(p)\overline{F}(1)+\cdots+\overline{G}(4)\overline{F}(p-3)\}+8a\overline{G}(3)F(p-3)\\&-4a\{\overline{G}(p+1)\overline{F}(1)+\cdots+\overline{G}(4)\overline{F}(p-2)\}+a[\overline{\overline{\overline{K}}}(p+1)\overline{F}(1)+\cdots\\&+\overline{\overline{\overline{K}}}(4)\overline{F}(p-2)].\end{aligned}$$

Using (6) again, we obtain

$4(W-V)=aE$. By Lemma 2, we have proved this lemma.

Lemma 4. *If* $\dfrac{\partial^2 F(n-1)}{\partial z^2}-aK*F(n-1)=0$ *then* $\dfrac{\partial^2 F(n)}{\partial z^2}-aK*F(n)=0.$

Proof. By the definition of the derivative, we have

$$\begin{aligned}\frac{\partial^2 F(n)}{\partial z^2}&=2\left(\frac{\partial F(n)}{\partial z}-\frac{\partial F(n-1)}{\partial z}\right)-\frac{\partial^2 F(n-1)}{\partial z^2}\\&=4\left\{F(n)-F(n-1)-\frac{\partial F(n-1)}{\partial z}\right\}-aK*F(n-1)=\cdots\cdots\cdots\cdots\\&=4\left\{F(n)-3F(n-1)+4F(n-2)-\cdots+(-1)^{n+1}4F(1)+(-1)^n 2F(0)\right.\\&\left.+(-1)^n\frac{\partial F(0)}{\partial z}\right\}-aK*F(n-1).\end{aligned}$$

From (8), we obtain $\dfrac{\partial^2 F(n)}{\partial z^2}-aK*F(n)=0.$ Thus, Lemma 4 is proved.

In conclusion, we have proved that (2) has a solution for the points on the positive x-axis. Also, we can prove that (2) has a solution for the points on the positive y-axis. By using similar process, we have that (2) has a solution $F(z)$ for the points on the real and imaginary axes. Following the remark of Duffin [6], a function $f\in A(R)$ is uniquely determined by its values on the real and imaginary axes. Therefore, Theorem 4 is proved.

3. Discrete derivative equations of the type $\dfrac{\partial^2 F(z)}{\partial z^2}-aK(z)*F(z)=b(z)$.

Theorem 5. *Let $K(z)$ be discrete analytic in R containing the origin. And if* $16-ah^3[K(0)+K(h)]\neq 0$ *for h equals to one of the values* ± 1 *or*

$\pm i$. *Then there exists a unique function* $F(z)$ *discrete analytic in* R, *such that*

(12) $\dfrac{\partial^2 F}{\partial z^2} - aK*F(z) = b(z)$ *with* $F(0) = C_1$ *and* $\dfrac{\partial F(0)}{\partial z} = C_2$, *where* $b(z) \in A(R)$. *And the solution of* (12) *can be calculated by the following stepping formula*:

(13) $$F(z+h) = \frac{1}{16 - ah^3[K(0)+K(h)]}\Big\{16[C_2(z+h)+C_1+H(z+h)] + ah^3[K(0)+K(h)]F(z) + 16a\int_0^z G(z+h-t) : F(t)\partial t\Big\}$$

with $\dfrac{\partial^2 F(0)}{\partial z^2} = b(0)$, *where* $H(z) = z*b(z)$ *and* $G(z) = z*K(z)$.

Proof. Let $M(z) = K*F(z)$, from (12) we have

$$F(z) = \int_0^z \int_0^{t_1} [aM(t)+b(t)]\partial t \partial t_1 + C_1 + C_2 z$$
$$= \int_0^z (z-t) : [aM(t)+b(t)]\partial t + C_2 z + C_1$$

i. e.

(14) $F(z) = C_2 z + C_1 + H(z) + aG*F(z)$.

This is a discrete Volterra integral equation. Since $K(z) \in A(R)$, $C_2 z + C_1 + H(z) \in A(R)$ and $16 - ah^3[K(0)+K(h)] \neq 0$ is equivalent to $ah[G(0) + G(h)] \neq 4$, and by Theorem 2 we obtain that there exists a unique discrete analytic solution $F(z)$ of (14). And the solution can be calculated by the following stepping formula.

$$F(z+h) = \frac{1}{4 - ah[G(0)+G(h)]}\Big\{4[C_2(z+h)+C_1+H(z+h)] + ah[G(0) + G(h)]F(z) + 4a\int_0^z G(z+h-t) : F(t)\partial t\Big\}$$

On the other hand, we can rewrite $F(z+h)$ into the form (13). Thus (13) is the required stepping formula for finding the unique solution $F(z)$ of (12). With the similar proof of Theorem 4, we see that the function $F(z)$ which is obtained uniquely from (13) is exactly a soluton of (12).

REFERENCES

[1] SHIH-TONG TU: Discrete analytic derivative equations of the first order. Math. J. Okayama Univ. Vol. 14, No. 2 (1970), 103—109.

[2] R. J. DUFFIN and C. S. DURIS: Discrete analytic continuation of solutions of difference equations. J. Math. anal. appl. 9. (1964), 252—267.
[3] J. FERRAND: Fonction préharmonique et fonctions préholomerphes. Bull. Sci. Math. 68 (1944), 152—180.
[4] S. HAYABARA: Operational calculus on the discrete analytic functions. Math. Japon 11 (1966), 35—65.
[5] R. J. DUFFIN and C. S. DURIS: A convolution product for discrete analytic function theory. Duke Math. J. 31 (1964), 199—220.
[6] R. J. DUFFIN: Basic properties of discrete analytic functions. Duke Math. J. 23 (1956), 335—363.

DEPARTMENT OF MATHEMATICS,
OKAYAMA UMIVERSITY

DEPARTMENT OF MATHEMATICS,
CHUNG YUAN CHRISTIAN COLLEGE OF SCIENCE AND ENGINEEING,
CHUNG LI, TAIWAN, CHINA

(*Received March 10, 1970*)