

Mathematical Journal of Okayama University

Volume 14, Issue 2

1969

Article 6

DECEMBER 1970

Some remarks on invariant subrings

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SOME REMARKS ON INVARIANT SUBRINGS

KAORU MOTOSE and HISAO TOMINAGA

In this paper, we shall treat with subrings of a ring which are setwise invariant relative to all inner automorphisms.

Throughout our study, we use the following conventions: U will represent a ring with 1, and B a subdirectly irreducible subring of U whose unique minimal ideal T is not nilpotent. If S is a ring with 1 then $\mathfrak{N}(S)$ and $(S)_A$ will mean the Jacobson radical of S and the ring of all row-finite matrices (x_{ij}) ($x_{ij} \in S, i, j \in A$), respectively. A unit s of S is called a *biregular element* if $1-s$ is also a unit. Following [4], the set of all biregular elements of S will be denoted by S^* , and S is defined to be *biregularly generated* (resp. *regularly generated*) if every element of S is a sum of biregular elements (resp. of units) in S . Finally, A will represent a unital subring of $U^{(1)}$ satisfying the following conditions:

(1) $A/\mathfrak{N}(A) \neq \text{GF}(2), (\text{GF}(2))_2$.

(2) A is (isomorphic to) $(D)_A$ with a local ring (or a completely primary ring in the sense of [5]) D . Here, $e_{\lambda\mu}$ will represent the matrix of A with 1 in the (λ, μ) -position and 0's elsewhere. We set $E = \{\sum_{\lambda, \mu} d_{\lambda\mu} e_{\lambda\mu} \mid d_{\lambda\mu} \in D\}$ (almost all $d_{\lambda\mu}$'s are 0), and $F = \sum_{\lambda} A e_{\lambda\lambda}$.

As to other notations and terminologies used in this paper, we follow the previous one [3].

1. In this section, we shall give a generalization of [3; Theorem 2].

Proposition 1. *If $B\tilde{A} \subset B^{(3)}$ and the right annihilator $r_A(T)$ of T in A is 0 then either $TE \subset T$ or $E \subset V_U(B)$.*

Proof. If a is a unit of A and $\{a, 1\}$ is not left B -free then $b_1 a - b_2 = 0$ with some $b_1 \neq 0, b_2 \in B$, and then, noting that T is the least non-zero ideal of B and $B\tilde{A} = B$, one will easily see $Ta \subset T$. Next, if a is biregular and $\{a, 1\}$ is left B -free, then for every $b \in B, ab = b'a$ and

1) A unital subring of U means a subring containing the identity element of U .

2) $\tilde{A}$ represents the multiplicative group of all inner automorphisms of U induced by units of A .

$(1-a)b=b''(1-a)$ ($b', b'' \in B$) yield $b=b''=b'$, whence it follows that a is contained in $V_v(B)$. Accordingly, we have $A^* \subset A_0 \cup V_v(B)$ where $A_0 = \{a \in A \mid Ta \subset T\}$. Now, we shall distinguish between two cases:

(1) $\#A=1$: Suppose that $A \not\subset V_v(B)$ and $A \not\subset A_0$. Then there exist some $a_1 \in A^* \setminus V_v(B)$ and $a_2 \in A^* \setminus A_0$. By the above remark, $a_1 \in A_0$, $a_2 \in V_v(B)$ and then we can easily see that $\{a_2, a_1\}$ is left B -free. If $a_1 + a_2$ is a unit of A then for every $b \in B$ $(a_1 + a_2)b = b'(a_1 + a_2)$ ($b' \in B$) and $a_2b = ba_2$ yield $(b' - b)a_2 + (b' - a_1ba_1^{-1})a_1 = 0$, which means $b' = b$. On the other hand, if $a_1 + a_2$ is not a unit then $1 - (a_1 + a_2) = (-a_1) \div (1 - a_2)$ is a unit of A and the above argument proves again $a_1 + a_2 \in V_v(B)$. But this contradicts $a_1 \notin V_v(B)$ and $a_2 \in V_v(B)$.

(2) $\#A > 1$: If R is an arbitrary subring of U with $R\tilde{A} \subset R$ and $Re_u \subset R$ for some $l \in A$ then $RE \subset R$. In fact, for every $\lambda \neq l$ and every $d \in D$ we obtain $(1 - de_{l\lambda})Re_u(1 - de_{l\lambda})^{-1} = R(e_u + de_{l\lambda})$ and $(1 + de_{\lambda l})Re_u(1 + de_{\lambda l})^{-1} = R(e_u + de_{\lambda l})$. Combining these with $Re_u \subset R$, we readily obtain $Rde_{l\lambda}, Rde_{\lambda l} \subset R$, whence it follows $RE \subset R$. If a_1 and a_2 are biregular elements of A such that $a_1a_2 \neq a_2a_1$ and $a_1 - a_2 = e_u$ for some $l \in A$, then by $r_A(T) = 0$ and $A^* \subset A_0 \cup V_v(B)$ we see that either (both a_1 and a_2 and hence) $a_1 - a_2 = e_u$ is in A_0 or in $V_v(B)$, and hence, recalling that $T\tilde{A} = T$ and $V_v(B)\tilde{A} = V_v(B)$, the remark stated just above proves that $E \subset A_0$ or $E \subset V_v(B)$. In what follows, we shall show that we can find such biregular elements a_1 and a_2 . To this end, we shall distinguish between four cases:

(i) D^* is non-empty and $\#A < \aleph_0$: Let d be an arbitrary element of D^* . Then, the following elements are requested ones:

$$a_1 = \begin{pmatrix} 1 & 1 & & \\ d & 0 & & \\ & & d & \\ & & & \ddots \\ & & & & d \end{pmatrix}, \quad a_2 = \begin{pmatrix} 0 & 1 & & \\ d & 0 & & \\ & & d & \\ & & & \ddots \\ & & & & d \end{pmatrix}.$$

(ii) D^* is non-empty and $\#A \geq \aleph_0$: One may regard $(D)_A$ as $((D)_2)_A$. Now, let d be an arbitrary element of D^* , and choose an arbitrary index $l \in A$. We set $a_1 = (x_{\lambda\mu})$ where $x_{ll} = \begin{pmatrix} 1 & 1 \\ d & 0 \end{pmatrix}$, $x_{\lambda\lambda} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ for $\lambda \neq l$ and $x_{\lambda\mu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $\lambda \neq \mu$, and $a_2 = (y_{\lambda\mu})$ where $y_{ll} = \begin{pmatrix} 0 & 1 \\ d & 0 \end{pmatrix}$, $y_{\lambda\lambda} = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix}$ for $\lambda \neq l$ and $y_{\lambda\mu} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ for $\lambda \neq \mu$. Then, a_1 and a_2 are elements requested.

(iii) D^* is empty (i. e. $D/\mathfrak{R}(D)=\text{GF}(2)$) and $2<\#A<\aleph_0$: In this case, we can take

$$a_1 = \begin{pmatrix} 1 & & & 1 \\ 1 & 1 & & \\ & \ddots & 0 & \\ & & \ddots & \\ & & & 1 & 0 \end{pmatrix} \quad \text{and} \quad a_2 = \begin{pmatrix} 0 & & & 1 \\ 1 & 1 & & \\ & \ddots & 0 & \\ & & \ddots & \\ & & & 1 & 0 \end{pmatrix}.$$

(iv) D^* is empty and $\#A \geq \aleph_0$: Let l be an arbitrary index in A .

Regarding $(D)_A$ as $((D)_3)_A$ we consider $a_1 = (x_{\lambda\mu})$ where $x_{ii} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$
 $x_{\lambda\lambda} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ for $\lambda \neq l$ and $x_{\lambda\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $\lambda \neq \mu$, and $a_2 = (y_{\lambda\mu})$ where
 $y_{\lambda\lambda} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $y_{\lambda\mu} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for $\lambda \neq \mu$. Then, a_1 and a_2 are elements requested.

Corollary 1. *Let a unital subring R of U be strongly primary (or primary, semi-perfect in the sense of [1]). If $B\bar{R} \subset B$ and $r_A(T)=0$ then either $TA \subset T$ or $A \subset V_v(B)$.*

The next contains evidently [3; Theorem 2].

Theorem 1. *Assume that $\mathfrak{R}(D)$ is left T -nilpotent (or right vanishing in the sense of [6]). If $B\bar{A} \subset B$ and $r_A(T)=0$ then either $TA \subset T$ or $A \subset V_v(B)$.*

Proof. By Proposition 1, $TE \subset T$ or $E \subset V_v(B)$. As $\mathfrak{R}(A) = (\mathfrak{R}(D))_A$ by [6; Theorem 1], [7; Theorem] enables us to see that A is regularly generated. Especially, every element of F is a sum of units in A and $T\bar{A} \subset T$ implies $TA = AT$. If u is an arbitrary unit of A then for every $\lambda \in A$ we can find an element $e_\lambda \in E$ such that $ue_{\lambda\lambda} = e_{\lambda\lambda}\tilde{u} \cdot u = e_{\lambda\lambda}\tilde{u} \cdot e_\lambda$, where $e_{\lambda\lambda}\tilde{u} = ue_{\lambda\lambda}u^{-1} \in F$. If $TE \subset T$ then $T\bar{A} \subset T$ implies $Tue_{\lambda\lambda} = T \cdot e_{\lambda\lambda}\tilde{u} \cdot e_\lambda = (Te_{\lambda\lambda})\tilde{u} \cdot e_\lambda \subset Te_\lambda \subset T$. Hence, $0 \neq TF = T \cdot \sum_\lambda Fe_{\lambda\lambda} \subset T$. Accordingly, we have $TFT = T$ and $TA = TFTA = TFAT = TFT = T$. On the other hand, if $E \subset V_v(B)$ then by $V_v(B)\bar{A} \subset V_v(B)$ we have $ue_{\lambda\lambda} = e_{\lambda\lambda}\tilde{u} \cdot e_\lambda \in V_v(B)$, which implies $F = \sum_\lambda Fe_{\lambda\lambda} \subset V_v(B)$. Now, for every unit a of A , $b \in B$ and $f \in F$ there holds $f(b - aba^{-1}) = fb - (fa)ba^{-1} = fb - b(fa)a^{-1} = 0$. Noting that $TF \neq 0$ implies $r_B(F) = 0$, we obtain $ba = ab$. Since A is regularly generated, the last means $A \subset V_v(B)$.

2. In this section, we shall give a slight improvement of [5; § 42] (cf. [2]).

Theorem 2. *Let S be a biregularly generated ring with 1, and R a two-sided simple unital subring of S . If $RS \subset R$ then $R=S$ or $R \subset V_S(S)$.*

Proof. As was noted in the proof of Proposition 1, $S^* \subset R \cup V_S(R)$. Hence, $S = R + V_S(R) = R \cdot V_S(R) = R \otimes_Z V_S(R)$, where Z is the center of R . Noting that Z is a field, one will readily see that $R=Z$ or $V_S(R)=Z$, namely, $S=V_S(R)$ or $S=R$.

The argument used in the proof of Theorem 1 suggests the following:

Theorem 3. *Let R be a two-sided simple unital subring of A . If $R\tilde{A} \subset R$ then either $R=A$ or $R \subset V_A(A)$.*

Proof. As evidently $r_A(R)=0$, Proposition 1 implies $E \subset R$ or $E \subset V_A(R)$. In case $\#A=1$, there is nothing to prove. Thus, in what follows, we may restrict our attention to the case $\#A>1$. Now, let $f=(a_{\lambda\mu})$ be an arbitrary element of F . Then there exists a finite subset I of A such that $\#I>1$ and $a_{\lambda\mu}=0$ for all $\mu \in A \setminus I$. By [7; Theorem], $f'=(a'_{\lambda\mu})$ ($\lambda, \mu \in I$) is a sum of two units in $(D)_I$: $f'=(a'_{\lambda\mu})+(a''_{\lambda\mu})$ ($\lambda, \mu \in I$). We consider here the elements $u=(u_{\lambda\mu})$ and $v=(v_{\lambda\mu})$ in A which are defined as follows:

$$u_{\lambda\mu} = \begin{cases} a'_{\lambda\mu} & \text{if } \lambda, \mu \in I \\ 1 & \text{if } \lambda = \mu \in A \setminus I \\ a_{\lambda\mu} & \text{if } \lambda \in A \setminus I \text{ and } \mu \in I \\ 0 & \text{elsewhere} \end{cases}$$

and

$$v_{\lambda\mu} = \begin{cases} a''_{\lambda\mu} & \text{if } \lambda, \mu \in I \\ -1 & \text{if } \lambda = \mu \in A \setminus I \\ 0 & \text{elsewhere} \end{cases}$$

To be easily seen, u and v are units of A and $f=u+v$. If $E \subset R$ then, as was noted in the proof of Theorem 1, there holds $RF \subset R$. Accordingly, $A=RA=RFRA=RF A=R$. On the other hand, if $E \subset V_A(R)$ then in the proof of Theorem 1 we have seen that $F \subset V_A(R)$. Moreover, if $f \in F$, $a \in A$ and $b \in R$ then $f(ba-ab)=fba-(fa)b=b(fa-fa)=0$. Noting that $r_A(F)=0$, it follows $ab=ba$. We have proved therefore $R \subset V_A(A)$.

Lemma 1. *Let S be a biregularly generated ring with 1, and R an artinian semi-simple subring of S . If $R\tilde{S} \subset R$ then $V_R(R) = R \cap V_S(S)$.*

Proof. Let $V_R(R) = Z_1 \oplus \cdots \oplus Z_t$, where Z_i is a field with the identity element e_i . Taking an arbitrary element $s \in S^*$, $se_1 = e_1s$ and $(1-s)e_1 = e_1(1-s)$ for some i, j , whence it follows $(e_i - e_j)s = e_1 - e_j$. If $e_1 \neq e_i$ then $e_i \neq e_j$ and hence $e_1s = e_i(e_i - e_j)s = e_i(e_1 - e_j) = 0$. This contradiction means $e_1\tilde{s} = e_1$. It follows therefore $Z_1\tilde{s} = Z_1\tilde{s} \cdot e_1\tilde{s} = Z_1\tilde{s} \cdot e_1$, namely, $Z_1\tilde{s} = Z_1$. Accordingly, for an arbitrary element $z \in Z_1$ there hold $sz = z's$ and $(1-s)z = z''(1-s)$ with some $z', z'' \in Z_1$, and so $(z' - z'')s = z - z''$. If $z \neq z'$ then $z' \neq z''$, and hence $se_1 = e_1s = (z' - z'')^{-1}(z - z'') \in Z_1$. But this implies a contradiction $z\tilde{s} = (e_1z)\tilde{s} = se_1 \cdot z \cdot s^{-1} = z \cdot e_1\tilde{s} = ze_1 = z$. We have seen therefore $sz = zs$, i. e. $Z_1 \subset V_S(s)$. Similarly, we have $Z_i \subset V_S(s)$ ($i = 1, \dots, t$). Hence, $V_R(R) \subset \bigcap_{s \in S^*} V_S(s) = V_S(S)$.

Theorem 4. *Let S be an artinian simple ring with 1 different from $(\text{GF}(2))_2$, and $R \neq 0$ a left perfect subring of S .³⁾ If $R\tilde{S} \subset R$ then either $R = S$ or $R \subset V_S(S)$.*

Proof. Noting that all the primitive idempotents of S are mutually conjugate with respect to inner automorphisms, we can easily see that R is a unital subring of S . In case $S = \text{GF}(2)$, there is nothing to prove. Thus, in below we may assume that S is biregularly generated ([4; Theorem]). As $R\tilde{S} \subset R$, we have $\mathfrak{N}(R)S = S\mathfrak{N}(R)$. Now, let s be an arbitrary element of $\mathfrak{N}(R)S$. Then $I = \{x - xs \mid x \in S\}$ is a left ideal of S and evidently $S = I + \mathfrak{N}(R)S$. By making use of the same argument as in [1; pp. 473—474], we can prove that $I = S$, namely, $\mathfrak{N}(R) \subset \mathfrak{N}(R)S \subset \mathfrak{N}(S) = 0$. Hence, R being simple by $V_R(R) = R \cap V_S(S)$ (Lemma 1), Theorem 2 proves that either $R = S$ or $R \subset V_S(S)$.

Evidently, Theorem 4 contains [5; Theorem 42.4] and yields Corollaries 42.5 and 42.6 of [5], which are stated as follows:

Corollary 2. *Let S be an artinian semi-simple ring: $S = S_1 \oplus \cdots \oplus S_t$, where each S_i is a simple ring different from $(\text{GF}(2))_2$. If R is a right artinian subring of S and $R\tilde{S} \subset R$ then $R = S_{i_1} \oplus \cdots \oplus S_{i_k} \oplus C'$ with suitable S_{i_j} 's and a subring C' of the center of S .*

Corollary 3. *Let S be a right artinian primary ring with 1 such*

3) A ring R with 1 is called a left perfect ring if $R/\mathfrak{N}(R)$ is artinian and $\mathfrak{N}(R)$ is left T -nilpotent (cf. [1]).

that $S/\mathfrak{R}(S) \neq (\text{GF}(2))_2$. If R is an artinian semi-simple subring of S and $R\tilde{S} \subset R$ then $\mathfrak{R}(S)$ is contained in $V_S(R)$ and either $S = R \oplus \mathfrak{R}(S)$ (as module) or $R \subset V_S(S)$.

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(Received March 31, 1970)