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A NOTE ON GALOIS EXTENSIONS OF DIVISION RINGS

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The purpose of this note is to prove a generalization of one of the theorems in Galois theory of commutative fields. Let a division ring P be Galois and locally finite¹⁾ over its subring ϕ . If A is a division subring of $V(V(\phi))$ containing ϕ such that it is a Galois extension of ϕ and if Σ is an arbitrary division subring of P containing ϕ , then ΣA , the division subring of P generated by Σ and A , is an outer Galois extension of Σ and its Galois group is isomorphic to that of $A/\Sigma \cap A$.

In § 1, it will be proved that, if a division ring P is locally finite over a division subring ϕ , then P is also locally finite over $V(V(\phi))$. In § 2, we shall prove that if P is Galois and locally finite over ϕ then P is also Galois over any division subring which is finite over ϕ . And finally in § 3, the above-mentioned theorem will be proved.

1. Throughout the present note, let P be a division ring and ϕ a division subring of P . For any non-zero element ρ of P , we denote by $\tilde{\rho}$ the inner automorphism induced by ρ : $\tilde{\rho} = \rho_i \rho_r^{-1}$. Similarly $\tilde{\Gamma}$ will mean the totality of inner automorphisms induced by non-zero elements of Γ where Γ is a subset of P . For a subring Σ of P , $(\tilde{\rho})_\Sigma$ and $\tilde{\Gamma}_\Sigma$ mean the restrictions of $\tilde{\rho}$ and $\tilde{\Gamma}$ onto Σ . $V(\Sigma)$ means as usual the centralizer of Σ in P . Then $\widetilde{V(\phi)}_\Sigma P_r$ is naturally a P_r -right module.

Lemma 1. *Let $\alpha_1, \dots, \alpha_n$ be non-zero elements of $V(\phi)$. Then $(\tilde{\alpha}_1)_\Sigma, \dots, (\tilde{\alpha}_n)_\Sigma$ are linearly right-independent over P_r if and only if $\alpha_1, \dots, \alpha_n$ are linearly right-independent over $V(\Sigma)$.*

Proof. If $\alpha_1, \dots, \alpha_n$ are linearly right-dependent over $V(\Sigma)$, we have a non-trivial relation $\sum_{i=1}^n \alpha_i \pi_i = 0$ with $\pi_i \in V(\Sigma)$. Then $\sum_{i=1}^n \tilde{\pi}_i \tilde{\alpha}_i \alpha_{ir} \pi_{ir} = 0$, where we put $\tilde{\pi}_i = 0$ in case $\pi_i = 0$. Since $(\tilde{\pi}_i)_\Sigma = 1$, we have a non-trivial relation $\sum_{i=1}^n (\tilde{\alpha}_i)_\Sigma (\alpha_i \pi_i)_r = 0$, which implies that $(\tilde{\alpha}_1)_\Sigma, \dots, (\tilde{\alpha}_n)_\Sigma$ are linearly right-dependent over P_r . Conversely suppose that $(\tilde{\alpha}_1)_\Sigma, \dots, (\tilde{\alpha}_n)_\Sigma$ are linearly right-dependent over P_r . Then we have their non-trivial relations and let one of the shortest relations among them be, for instance,

¹⁾ As to notations and terminologies used in this note we follow [3] and [4].

$\sum_{i=1}^s (\tilde{\alpha}_i)_\Sigma \rho_{ir} = 0$ with non-zero $\rho_i \in P$. From this we have $\sum_{i=1}^s (\alpha_{ir})_\Sigma \rho'_{ir} = 0$ where $\rho'_i = \alpha_i^{-1} \rho_i$. In the above relation we may assume $\rho'_1 = 1$ from the beginning. Then we shall show that each ρ'_i is in $V(\Sigma)$. For, if not, $\rho'_j \notin V(\Sigma)$ for some j , that is, there exists an element σ of Σ such that $\sigma \rho'_j \neq \rho'_j \sigma$. Clearly we have $\sum_{i=1}^s (\sigma_r(\alpha_{ir}))_\Sigma \rho'_{ir} - \sum_{i=1}^s (\alpha_{ir})_\Sigma \rho'_{ir} \sigma_r = 0$, whence we have $\sum_{i=2}^s (\alpha_{ir})_\Sigma (\sigma_r \rho'_{ir} - \rho'_{ir} \sigma_r) = 0$. Thus we obtain a shorter non-trivial relation $\sum_{i=2}^s (\alpha_{ir})_\Sigma \theta_{ir} = 0$ with $\theta_i = \sigma \rho'_i - \rho'_i \sigma$, being a contradiction. Accordingly we have shown $\rho'_i \in V(\Sigma)$. Then $0 = \sum_{i=1}^s (\alpha_{ir} \rho'_{ir})_\Sigma = \sum_{i=1}^s (\rho'_{ir} \alpha_{ir})_\Sigma$, that is, $\sum_{i=1}^s \alpha_i \rho'_i = 0$ with $\rho'_i \in V(\Sigma)$, which completes our proof.

Lemma 2. *Let Σ be a subring of P containing ϕ . Then $[\Sigma : \phi]_i \geq [V(\phi) : V(\Sigma)]_r^{2)}$. Moreover, if $V(V(\phi)) = \phi$, equality holds in the above relation.*

Proof. Let $\mathfrak{M}$ be the set of all homomorphisms of ϕ -module Σ into P . Then $\mathfrak{M}$ is a P_r -right module and $[\Sigma : \phi]_i = [\mathfrak{M} : P_r]_r$. Clearly $\mathfrak{M} \supseteq \widetilde{V(\phi)_\Sigma P_r}$. Since $[\widetilde{V(\phi)_\Sigma P_r} : P_r]_r = [V(\phi) : V(\Sigma)]_r$ by Lemma 1, we have $[\Sigma : \phi]_i \geq [V(\phi) : V(\Sigma)]_r$. If, moreover, $V(V(\phi)) = \phi$, $\mathfrak{M}$ is the topological closure of $\widetilde{V(\phi)_\Sigma P_r}$ by Jacobson's density theorem [1, p. 31]. Then $[\Sigma : \phi]_i = [\mathfrak{M} : P_r]_r = [\widetilde{V(\phi)_\Sigma P_r} : P_r]_r = [V(\phi) : V(\Sigma)]_r$.

Theorem 1. *If P is locally finite over ϕ , then P is also locally finite over $V(V(\phi))$.*

Proof. Let ϕ_0 be $V(V(\phi))$ and $\phi_0(\alpha_1, \dots, \alpha_n)$ a subring generated by ϕ_0 and a finite number of elements $\alpha_1, \dots, \alpha_n$ of P . Then $\infty > [\phi(\alpha_1, \dots, \alpha_n) : \phi]_i \geq [V(\phi) : V(\phi(\alpha_1, \dots, \alpha_n))]_r = [V(V(\phi(\alpha_1, \dots, \alpha_n))) : V(V(\phi))]_i = [\phi_0(\alpha_1, \dots, \alpha_n) : \phi_0]_i$ by Lemma 2.³⁾

2. When a subring ϕ of P is the fixring of an automorphism group of P , that is, when ϕ consists of all the elements left invariant by an automorphism group of P , we say that P is Galois over ϕ or P/ϕ is Galois.

2) Provided that we do not distinguish between two infinite dimensions.

3) Note that $V(\phi(\alpha_1, \dots, \alpha_n)) = V(\phi_0(\alpha_1, \dots, \alpha_n))$ and $V(V(\phi(\alpha_1, \dots, \alpha_n))) \supseteq \phi_0(\alpha_1, \dots, \alpha_n)$.

Theorem 2. *Let P/ϕ be locally finite and Galois. Then P/Σ is Galois for each subring Σ of P containing ϕ which is finite over ϕ .*

Proof. We may assume here $P \neq \Sigma$. Let ρ be an arbitrary element of P not contained in Σ , and Σ' a subring of P generated by Σ and ρ . We denote by $\mathfrak{M}'$ the set of all homomorphisms of ϕ_i -module Σ' into P . $\mathfrak{M}'$ is a Σ'_r - P_r two-sided module. Then, by Jacobson's density theorem, $\mathfrak{M}' = \mathfrak{G}_{\Sigma'} P_r$, where $\mathfrak{G}$ is a regular automorphism group of P/ϕ . Similarly let $\mathfrak{M}$ be the set of all homomorphisms of Σ_i -module Σ' into P . Clearly $\mathfrak{M}' \supseteq \mathfrak{M}$. Since $\mathfrak{M}'$ is a completely reducible Σ'_r - P_r two-sided module, $\mathfrak{M}$ is also a completely reducible Σ'_r - P_r two-sided module. Now we shall show that $\mathfrak{M} = (\mathfrak{G}_{\Sigma'} \cap \mathfrak{M}) P_r$. Suppose, on the contrary, $\mathfrak{M} \neq (\mathfrak{G}_{\Sigma'} \cap \mathfrak{M}) P_r$. Then $\mathfrak{M}$ contains an irreducible Σ'_r - P_r two-sided submodule $\mathfrak{N}$ which is not wholly contained in $(\mathfrak{G}_{\Sigma'} \cap \mathfrak{M}) P_r$. As $\mathfrak{N}$ is contained in $\mathfrak{M}' = \mathfrak{G}_{\Sigma'} P_r$, a similar argument as in the proof of [4, Lemma 3] proves that $\mathfrak{N} = T_{\Sigma'} P_r$ where T is an element of $\mathfrak{G}$. This implies that $\mathfrak{N} = T_{\Sigma'} P_r \subseteq (\mathfrak{G}_{\Sigma'} \cap \mathfrak{M}) P_r$, which is a contradiction. Thus, setting $\mathfrak{H}_{\Sigma'} = \mathfrak{G}_{\Sigma'} \cap \mathfrak{M}$ with a subgroup $\mathfrak{H}$ of $\mathfrak{G}$, we have $\mathfrak{M} = \mathfrak{H}_{\Sigma'} P_r$. Naturally $\mathfrak{H}$ is identical on Σ . We shall show $\rho \mathfrak{H} \neq \rho$. In fact, $\rho \mathfrak{H} = \rho$ implies $\mathfrak{H}_{\Sigma'} = 1$, that is, $\mathfrak{M} = 1 P_r$. But this contradicts $[\mathfrak{M} : P_r]_r = [\Sigma' : \Sigma]_i > 1$. Since ρ is an arbitrary element of P not contained in Σ , we have proved P/Σ is Galois.

As is easily seen from the above proof, we may restate Theorem 2 in the following way.

Theorem 2'. *Let P/ϕ be Galois and locally finite, and let $\mathfrak{G}$ be a regular automorphism group of P/ϕ . If Σ is an intermediate subring of P/ϕ with $[\Sigma : \phi]_i < \infty$ then there exists a subgroup $\mathfrak{H}$ of $\mathfrak{G}$ such that Σ is the fixing of $\mathfrak{H}$.*

We may remark here the following: Let P/ϕ be Galois. Then Theorem 2 shows that the assumptions (a) — (d) introduced in [2; §3] are fulfilled when and only when P is locally finite over ϕ and $[V(\phi) : V(P)] < \infty$.

3. For subrings Σ and Γ of P , we denote $V(V(\Sigma))$ by Σ_0 and denote by $\Sigma\Gamma$ the subring of P generated by Σ and Γ .

Lemma 3. *Let P/ϕ be Galois and locally finite and let Σ be a subring of P containing ϕ such that $[\Sigma\phi_0 : \phi_0]_i < \infty$. Then $\Sigma_0 = \Sigma\phi_0$, and Σ_0/Σ is Galois and locally finite with Galois group which is isomorphic to that of $\phi_0/\phi_0 \cap \Sigma$.*

Proof. Since $V(\Sigma_0) = V(\Sigma\phi_0)$, we obtain $\Sigma_0 \subseteq V(V(\Sigma\phi_0))$ and

Lemma 2 shows $\infty > [\Sigma\phi_0 : \phi_0]_t = [V(\phi_0) : V(\Sigma\phi_0)]_r = [V(\phi_0) : V(\Sigma_0)]_r = [\Sigma_0 : \phi_0]_t$. Hence we have $\Sigma_0 = \Sigma\phi_0 = \phi_0(\alpha_1, \dots, \alpha_n)$ with some $\alpha_1, \dots, \alpha_n$ of Σ . Then $P/\phi(\alpha_1, \dots, \alpha_n)$ is Galois by Lemma 2, further $V(V(\phi(\alpha_1, \dots, \alpha_n))) = \Sigma_0$ implies that $\Sigma_0/\phi(\alpha_1, \dots, \alpha_n)$ is and hence Σ_0/Σ is outer Galois.⁴⁾ Since the Galois group $\mathfrak{H}'$ of $\Sigma_0/\phi(\alpha_1, \dots, \alpha_n)$ is locally finite, that of Σ_0/Σ is so, whence Σ_0 is locally finite over Σ .⁵⁾ Further, noting that $\mathfrak{H}'$ is the topological closure of $\mathfrak{G}'_{\Sigma_0}$ where $\mathfrak{G}'$ is the Galois group of $P/\phi(\alpha_1, \dots, \alpha_n)$ [4, Theorem 4], we readily see that $\phi_0 \mathfrak{H}' = \phi_0$. In virtue of this fact, we shall prove that the Galois group $\mathfrak{H}$ of Σ_0/Σ is isomorphic with that of $\phi_0/\phi_0 \cap \Sigma$. Evidently $\mathfrak{H}_{\phi_0}$ is an automorphism group of $\phi_0/\phi_0 \cap \Sigma$, and so $\mathfrak{H}_{\phi_0}$ is dense in the Galois group $\mathfrak{H}_0$ of $\phi_0/\phi_0 \cap \Sigma$ by [4, Theorem 4]. Moreover, recalling that $\Sigma_0 = \phi_0(\alpha_1, \dots, \alpha_n)$, it is easy to see that $T \rightarrow T_{\phi_0}$ ($T \in \mathfrak{H}$) is a continuous isomorphism of the compact group $\mathfrak{H}$ into the compact group $\mathfrak{H}_0$. Hence $\mathfrak{H}$ is isomorphic to $\mathfrak{H}_{\phi_0} = \mathfrak{H}_0$.

Lemma 4. *Let P/ϕ be Galois and locally finite. Then, for any subring Σ of P containing ϕ , $\phi_0\Sigma/\Sigma$ is outer Galois and locally finite with the Galois group isomorphic with that of $\phi_0/\phi_0 \cap \Sigma$.*

Proof. Since P/ϕ_0 is locally finite by Theorem 1, $\phi_0\Sigma = \bigcup_{\nu} \Gamma_{\nu}$ where Γ_{ν} are all the subrings of the form $\Gamma_{\nu} = \phi_0(\sigma_1, \dots, \sigma_n)$ for some $\sigma_i \in \Sigma$. Put $\Sigma_{\nu} = \Gamma_{\nu} \cap \Sigma$. Then it is easy to see that $\Gamma_{\nu} = \phi_0\Sigma_{\nu}$ and $\Sigma_{\nu} \cap \phi_0 = \Sigma \cap \phi_0$. If T is an element of the Galois group $\mathfrak{H}_0$ of $\phi_0/\phi_0 \cap \Sigma$ then, as is seen from the proof of Lemma 3, T can be uniquely extended to an element $T^{(\nu)}$ of the Galois group $\mathfrak{H}^{(\nu)}$ of $\Gamma_{\nu}/\Sigma_{\nu}$ and $\mathfrak{H}_{\phi_0}^{(\nu)} = \mathfrak{H}_0$. Now if $\Gamma_{\nu} \subseteq \Gamma_{\mu}$ then $\Gamma_{\nu}T^{(\mu)} = (\phi_0T)(\Sigma_{\nu}T^{(\mu)}) = \phi_0\Sigma_{\nu} = \Gamma_{\nu}$, that is, $T_{\Gamma_{\nu}}^{(\mu)} = T^{(\nu)}$. Thus we can define an automorphism $T^{(0)}$ of $\phi_0\Sigma$ in the following way: $\rho T^{(0)} = \rho T^{(\nu)}$ if $\rho \in \Gamma_{\nu}$. We denote here by $\mathfrak{G}'$ the totality of these extended automorphisms of automorphisms in $\mathfrak{H}_0$. Then evidently $(\phi_0\Sigma)\mathfrak{G}' = (\bigcup_{\nu} \Gamma_{\nu})\mathfrak{G}' = \bigcup_{\nu} \Gamma_{\nu} = \phi_0\Sigma$ and the fixing of $\mathfrak{G}'$ is $\bigcup_{\nu} \Sigma_{\nu} = \Sigma$. Clearly $V(\Sigma) = V(\phi_0\Sigma)$, and hence $\phi_0\Sigma/\Sigma$ is outer Galois. Noting that $\mathfrak{G}'$ is identical on Σ , we readily see that $\mathfrak{G}'$ is a locally finite group of $\phi_0\Sigma/\Sigma$, whence $\phi_0\Sigma/\Sigma$ is locally finite [4, p.43] and the Galois group $\mathfrak{G}$ of $\phi_0\Sigma/\Sigma$ is the topological closure of $\mathfrak{G}'$ [4, Theorem 4]. Accordingly $\phi_0\mathfrak{G} = \phi_0$ and $\Gamma_{\nu}\mathfrak{G} = \Gamma_{\nu}$, and so our assertion will be easily seen by considering the mapping $S \rightarrow S_{\phi_0}$ ($S \in \mathfrak{G}$). (In fact, $\mathfrak{G}'$ coincides with $\mathfrak{G}$.)

⁴⁾ See [1, Proposition 7.6.3]. Cf. also [2], [3] and [4].

⁵⁾ See [4, p.43].

Finally we shall prove the following :

Theorem 3. *Let P/Φ be Galois and locally finite. If A is a subring of Φ_0 containing Φ , and A/Φ is Galois and if Σ is any subring of P containing Φ , then $A\Sigma/\Sigma$ is outer Galois and locally finite with the Galois group which is isomorphic to that of $A/A \cap \Sigma$.*

Proof. Noting that A is normal over Φ as a subring of Φ_0 by [4, Theorem 5], our assertion will be easily seen from the proof of Lemma 4.

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