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Ryo Takahashi*

*Okayama University

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NAGATA CRITERION FOR SERRE'S (R_n) AND (S_n) -CONDITIONS

RYO TAKAHASHI*

1. INTRODUCTION

Throughout the present paper, we assume that all rings are noetherian commutative rings.

First of all, we recall Serre's (R_n) and (S_n) -conditions for a ring A . These are defined as follows. Let n be an integer.

(R_n) : If $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\operatorname{ht}(\mathfrak{p}) \leq n$, then $A_{\mathfrak{p}}$ is regular.

(S_n) : $\operatorname{depth}(A_{\mathfrak{p}}) \geq \inf(n, \operatorname{ht}(\mathfrak{p}))$ for all $\mathfrak{p} \in \operatorname{Spec}(A)$.

Let $\mathbb{P}$ be a property of local rings. For a ring A we put

$$\mathbb{P}(A) = \{\mathfrak{p} \in \operatorname{Spec}(A) \mid \mathbb{P} \text{ holds for } A_{\mathfrak{p}}\}$$

and call it the $\mathbb{P}$ -locus of A . The following statement is called the (ring-theoretic) Nagata criterion for the property $\mathbb{P}$, and we abbreviate it to (NC).

(NC) : If A is a ring and if $\mathbb{P}(A/\mathfrak{p})$ contains a non-empty open subset of $\operatorname{Spec}(A/\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Spec}(A)$, then $\mathbb{P}(A)$ is open in $\operatorname{Spec}(A)$.

This statement was invented by Nagata in 1959. In algebraic geometry, there is a problem asking when the regular locus (that is, the non-singular locus) of a ring is open. He proposed the above criterion to consider this problem, and he proved that (NC) holds for $\mathbb{P} = \text{regular}$ ([6]). There are some other properties $\mathbb{P}$ for which (NC) holds, for example, $\mathbb{P} = \text{Cohen-Macaulay}$ ([3], [4]), Gorenstein ([2], [4]), and complete intersection ([2]). On the other hand, it is easy to see that (NC) holds for $\mathbb{P} = (\text{integral domain, coprimary (a ring } A \text{ is called coprimary if } \sharp \operatorname{Ass}(A) = 1), (R_0), (S_1), \text{ reduced, and normal})$. Moreover, as corollaries of these results, we easily see that the following proposition is true for $\mathbb{P} = \text{Cohen-Macaulay}$ ([3], [4]), Gorenstein ([4]), domain, coprimary, (R_0) , (S_1) , and reduced.

Let $\mathbb{P}$ be a property for which (NC) holds. Then, for a ring A satisfying $\mathbb{P}$, the $\mathbb{P}$ -locus of a homomorphic image of A is open.

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It is known that the properties “regular”, “Cohen-Macaulay”, “reduced”, and “normal” are described by using (R_n) and (S_n) . Since (NC) holds for each of these properties, we naturally expect that (NC) may hold for (R_n) and (S_n) for every $n \geq 0$. This is in fact true, and the main purpose of this paper is to give its complete proof.

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2. (NC) FOR (S_n) -CONDITION

The following lemma should be referred to [3] §22.

Lemma 2.1. *Let A be a domain, B an A -algebra of finite type, and M a finite B -module. Then there exists $f (\neq 0) \in A$ such that M_f is A_f -free (where A_f is the localization of A with respect to the multiplicatively closed set $\{1, f, f^2, \dots\}$).*

Now we can prove the main result of this section.

Theorem 2.2. (NC) holds for $\mathbb{P} = (S_n)$.

Proof. We prove the theorem by induction on n . It is easy to see that (NC) holds for $\mathbb{P} = (S_0)$ and (S_1) respectively, hence we assume $n \geq 2$ in the rest. Suppose that a ring A satisfies the assumption in (NC). We want to prove that the locus $S_n(A)$ is open in $\text{Spec}(A)$. Since (S_n) implies (S_{n-1}) , the locus $S_{n-1}(A)$ is open in $\text{Spec}(A)$ by induction hypothesis. Therefore we can write $S_{n-1}(A) = \bigcup_{i=1}^s D(f_i)$ with $f_i \in A$, hence $S_n(A) = \bigcup_{i=1}^s (S_n(A) \cap D(f_i)) = \bigcup_{i=1}^s S_n(A_{f_i})$. Since $S_{n-1}(A_{f_i}) = S_{n-1}(A) \cap D(f_i) = D(f_i) = \text{Spec}(A_{f_i})$, the condition (S_{n-1}) holds for A_{f_i} . Thus, replacing A by A_{f_i} , to prove the openness of $S_n(A)$ we may assume that

(*) the condition (S_{n-1}) holds for A .

Put $\mathcal{I} = \{I \mid I \text{ is an ideal of } A \text{ and } S_n(A)^c \subseteq V(I)\}$, where $S_n(A)^c$ is the complement set $\text{Spec}(A) - S_n(A)$. We have $\mathcal{I} \neq \emptyset$ because $(0) \in \mathcal{I}$. Since A is noetherian, $\mathcal{I}$ has maximal elements. Let I be one of them. If $I = A$ then $S_n(A) = \text{Spec}(A)$ which is open in $\text{Spec}(A)$. Therefore we assume that $I \subsetneq A$. It is easy to see from the maximality that $\sqrt{I} = I$ and that $S_n(A)^c = V(I)$. It follows from this that I has a primary decomposition of the form $I = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_t$, where each $\mathfrak{p}_i$ is a prime ideal, and we may assume that there are no inclusion relations between the $\mathfrak{p}_i$'s and that $\text{ht}(\mathfrak{p}_1) \leq \text{ht}(\mathfrak{p}_i)$ for all i .

Now we claim that

$$(1) \text{ht}(I) \geq n,$$

- (2) $\mathfrak{p}_i \in S_n(A)^c$ for all i ,
 (3) $S_n(A)^c = V(I)$.

It follows from (3) that $S_n(A) = D(I)$, which shows that $S_n(A)$ is open in $\text{Spec}(A)$, proving the theorem. We prove these in turn.

(1) It suffices to prove that $\text{ht}(\mathfrak{p}_1) \geq n$. To prove this by contradiction, suppose that $l := \text{ht}(\mathfrak{p}_1) \leq n - 1$. By (*) we get $\text{depth}(A_{\mathfrak{p}_1}) \geq \inf(n - 1, \text{ht}(\mathfrak{p}_1)) = \text{ht}(\mathfrak{p}_1) = l$, hence there exist $c_i \in \mathfrak{p}_1$ and $f \in A - \mathfrak{p}_1$ such that $c_1, \dots, c_l$ is an A_f -sequence in $\mathfrak{p}_1 A_f$ and that $(c_1, \dots, c_l)A_f$ is $\mathfrak{p}_1 A_f$ -primary. Now we can take $g \in \bigcap_{i=2}^l \mathfrak{p}_i - \mathfrak{p}_1$ such that $IA_g = \mathfrak{p}_1 A_g$ because $\mathfrak{p}_i \not\subseteq \mathfrak{p}_1$ for all $i \geq 2$. Moreover, by the assumption in (NC), there exists $h \in A - \mathfrak{p}_1$ such that $D(h) \cap V(\mathfrak{p}_1) \subseteq S_n(A/\mathfrak{p}_1)$, hence the condition (S_n) holds for $A_h/\mathfrak{p}_1 A_h$. Put $x = fgh \in A - \mathfrak{p}_1$. Replacing A by A_x , we may assume that

$$\begin{cases} c_1, \dots, c_l \text{ is an } A\text{-sequence in } \mathfrak{p}_1, \\ (c_1, \dots, c_l) \text{ is } \mathfrak{p}_1\text{-primary (hence } \mathfrak{p}_1^r \subseteq (\underline{c}) \text{ for some } r \in \mathbf{N}), \\ I = \mathfrak{p}_1 \text{ (hence } \overline{S_n(A)^c} = V(\mathfrak{p}_1)), \\ (S_n) \text{ holds for } A/\mathfrak{p}_1. \end{cases}$$

Moreover, by Lemma 2.1, replacing A by A_y with some $y \in A - \mathfrak{p}_1$, we may assume that

$$\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1} + (\underline{c}) \cap \mathfrak{p}_1^i \text{ is } A/\mathfrak{p}_1\text{-free } (1 \leq i < r).$$

Now note that $S_n(A)^c \neq \emptyset$. In fact, if $S_n(A)^c = \emptyset$ then $V(\mathfrak{p}_1) = \overline{S_n(A)^c} = \emptyset$ hence $\mathfrak{p}_1 = A$, a contradiction. Therefore we have $S_n(A)^c \neq \emptyset$. We would like to prove that $A_{\mathfrak{p}}$ satisfies the condition (S_n) for any $\mathfrak{p} \in S_n(A)^c$. If this is true, then we have a contradiction since $\mathfrak{p} \notin S_n(A)$. Therefore, we will have $\text{ht}(\mathfrak{p}_1) \geq n$ as desired. To prove that (S_n) holds for $A_{\mathfrak{p}}$, take $\mathfrak{p}' \in \text{Spec}(A)$ with $\mathfrak{p}' \subseteq \mathfrak{p}$, and $\mathfrak{p}'' \in V(\mathfrak{p}' + \mathfrak{p}_1)$ such that $\text{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) = \text{ht}(\mathfrak{p}''/\mathfrak{p}_1)$. (Since $\mathfrak{p}', \mathfrak{p}_1 \subseteq \mathfrak{p}$, we have $V(\mathfrak{p}' + \mathfrak{p}_1) \neq \emptyset$.) We should divide the proof into two cases.

i) The case when $\text{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \leq n$:

Since $\text{ht}(\mathfrak{p}''/\mathfrak{p}_1) \leq n$, $A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''} = (A/\mathfrak{p}_1)_{\mathfrak{p}''/\mathfrak{p}_1}$ is CM. Replacing A by $A/(\underline{c})$, we may assume that $\mathfrak{p}_1^r = (0)$ and that $\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1}$ is $A/\mathfrak{p}_1$ -free. Therefore, $\text{depth}(A_{\mathfrak{p}''}) = \text{depth}(A_{\mathfrak{p}''}/\mathfrak{p}_1^r A_{\mathfrak{p}''}) = \text{depth}(A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''}) = \text{ht}(\mathfrak{p}''/\mathfrak{p}_1) = \text{ht}(\mathfrak{p}'')$, hence $A_{\mathfrak{p}''}$ is CM. It follows that $A_{\mathfrak{p}'} = (A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$ is CM.

ii) The case when $\text{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \geq n$:

Let $\mathfrak{q}/\mathfrak{p}_1 \in V(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)$. Then $\text{ht}(\mathfrak{q}/\mathfrak{p}_1) \geq n$, hence $\text{depth}((A/\mathfrak{p}_1)_{\mathfrak{q}/\mathfrak{p}_1}) \geq n$. Thus, $\text{depth}_{\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1}(A/\mathfrak{p}_1) \geq n$. Therefore there exist $c'_i \in \mathfrak{p}'$ such that

$$c'_1, \dots, c'_n \text{ is an } A/\mathfrak{p}_1\text{-sequence in } \mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1.$$

Since $\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1} + (\underline{c}) \cap \mathfrak{p}_1^i$ is $A/\mathfrak{p}_1$ -free, one can show that

$$c'_1, \dots, c'_n \text{ is an } A/(\underline{c})\text{-sequence in } \mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1.$$

Hence $c_1, \dots, c_l, c'_1, \dots, c'_n$ is an A -sequence in $\mathfrak{p}''$, so an $A_{\mathfrak{p}''}$ -sequence in $\mathfrak{p}''A_{\mathfrak{p}''}$. Therefore,

$$c'_1, \dots, c'_n, c_1, \dots, c_l \text{ is an } A_{\mathfrak{p}''}\text{-sequence in } \mathfrak{p}''A_{\mathfrak{p}''}.$$

Hence $c'_1, \dots, c'_n$ is an $A_{\mathfrak{p}''}$ -sequence in $\mathfrak{p}'A_{\mathfrak{p}''}$, so an $A_{\mathfrak{p}'} = (A_{\mathfrak{p}''})_{\mathfrak{p}'A_{\mathfrak{p}''}}$ -sequence in $\mathfrak{p}'A_{\mathfrak{p}'} = \mathfrak{p}'(A_{\mathfrak{p}''})_{\mathfrak{p}'A_{\mathfrak{p}''}}$. It follows that $\text{depth}(A_{\mathfrak{p}'}) \geq n$.

As we have remarked above, it follows from *i*), *ii*) that $\text{ht}(\mathfrak{p}_1) \geq n$.

(2) To prove it by contradiction, suppose that $\mathfrak{p}_k \in S_n(A)$ for some k . Since $I \subseteq \mathfrak{p}_k$, we have $\text{ht}(\mathfrak{p}_k) \geq n$, hence $\text{depth}(A_{\mathfrak{p}_k}) \geq \inf(n, \text{ht}(\mathfrak{p}_k)) = n$. Therefore, there exist $c_i \in \mathfrak{p}_k$ and $f \in A - \mathfrak{p}_k$ such that $c_1, \dots, c_n$ is an A_f -sequence in $\mathfrak{p}_kA_f$ and that $IA_f = \mathfrak{p}_kA_f$. Since $\mathfrak{p}_k \in V(I) = \overline{S_n(A)^c}$, we have $D(f) \cap S_n(A)^c \neq \emptyset$. Let $\mathfrak{p}$ be a minimal element of this set. Since $\mathfrak{p} \in S_n(A)^c \subseteq V(I)$, we have $I \subseteq \mathfrak{p}$, hence $\mathfrak{p}A_f \supseteq IA_f = \mathfrak{p}_kA_f$. Therefore $c_1, \dots, c_n$ is an A_f -sequence in $\mathfrak{p}A_f$, hence is an $A_{\mathfrak{p}} = (A_f)_{\mathfrak{p}A_f}$ -sequence in $\mathfrak{p}A_{\mathfrak{p}}$. It follows that $\text{depth}(A_{\mathfrak{p}}) \geq n = \inf(n, \text{ht}(\mathfrak{p}))$. On the other hand, if $\mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p}' \subsetneq \mathfrak{p}$, then we have $\mathfrak{p}' \notin D(f) \cap S_n(A)^c$ by the minimality of $\mathfrak{p}$. Since $\mathfrak{p} \in D(f)$, we have $\mathfrak{p}' \in D(f)$. Therefore we have $\mathfrak{p}' \notin S_n(A)^c$, hence (S_n) holds for $A_{\mathfrak{p}'}$. Thus, we see that (S_n) holds for $A_{\mathfrak{p}}$, contrary to the choice of $\mathfrak{p}$.

(3) We have $S_n(A)^c \subseteq \overline{S_n(A)^c} = V(I)$. Suppose that $S_n(A)^c \subsetneq V(I)$. Then there exists $\mathfrak{p} \in V(I)$ such that $\mathfrak{p} \notin S_n(A)^c$. Hence we have $\mathfrak{p}_k \subseteq \mathfrak{p}$ for some k and $\mathfrak{p} \in S_n(A)$. Therefore (S_n) holds for $(A_{\mathfrak{p}})_{\mathfrak{p}_kA_{\mathfrak{p}}} = A_{\mathfrak{p}_k}$. It follows that $\mathfrak{p}_k \in S_n(A)$, contrary to (2). $\square$

3. (NC) FOR (R_n) -CONDITION

Consider the following condition. Let n be an integer and let A be a local ring.

(R'_n) : If $\mathfrak{p} \in \text{Spec}(A)$ and $\text{codim}(\mathfrak{p}) \leq n$, then $A_{\mathfrak{p}}$ is regular.

Here the codimension of an ideal I of A is defined as follows.

$$\text{codim}(I) = \dim(A) - \dim(A/I).$$

Lemma 3.1. *Let A be a local ring. Then (R_n) holds for A if and only if (R'_n) holds for $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(A)$.*

Proof. Suppose that (R_n) holds for A . Let $\mathfrak{p} \in \text{Spec}(A)$, and let $\mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and that $\text{codim}(\mathfrak{p}'A_{\mathfrak{p}}) \leq n$. Then we have $\text{ht}(\mathfrak{p}'A_{\mathfrak{p}}) \leq \text{codim}(\mathfrak{p}'A_{\mathfrak{p}}) \leq n$, hence $(A_{\mathfrak{p}})_{\mathfrak{p}'A_{\mathfrak{p}}}$ is regular since (R_n) holds for $A_{\mathfrak{p}}$. It follows that (R'_n) holds for $A_{\mathfrak{p}}$. Conversely, suppose that (R'_n) holds for $A_{\mathfrak{p}}$ for every $\mathfrak{p} \in \text{Spec}(A)$. Let $\mathfrak{p} \in \text{Spec}(A)$, and let $\mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and that $\text{ht}(\mathfrak{p}'A_{\mathfrak{p}}) \leq n$. Then we have $\text{codim}(\mathfrak{p}'A_{\mathfrak{p}'}) = \text{ht}(\mathfrak{p}') = \text{ht}(\mathfrak{p}'A_{\mathfrak{p}}) \leq n$, hence $(A_{\mathfrak{p}})_{\mathfrak{p}'A_{\mathfrak{p}}} = A_{\mathfrak{p}'} = (A_{\mathfrak{p}'})_{\mathfrak{p}'A_{\mathfrak{p}'}}$ is regular. It follows that (R_n) holds for $A_{\mathfrak{p}}$. Therefore, (R_n) holds for A . $\square$

The following theorem is the main result of this section.

Theorem 3.2. (NC) holds for $\mathbb{P} = (R_n)$.

Proof. We prove this theorem by induction on n . It is easy to see that (NC) holds for $\mathbb{P} = (R_0)$, hence we assume $n \geq 1$ in the rest. We discuss in the same way as the proof of Theorem 2.2. Suppose that a ring A satisfies the assumption in (NC). Let I be one of the maximal elements of the set $\{I \mid I \text{ is an ideal of } A \text{ and } R_n(A)^c \subseteq V(I)\}$. We may assume that

$$\left\{ \begin{array}{l} (R_{n-1}) \text{ holds for } A \cdots (*), \\ I \subsetneq A, \\ \sqrt{I} = I, \\ \overline{R_n(A)^c} = V(I), \\ I = \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t \text{ (with some } \mathfrak{p}_i \in \text{Spec}(A)), \\ \text{there are no inclusion relations between the } \mathfrak{p}_i\text{'s,} \\ \text{ht}(\mathfrak{p}_1) \leq \text{ht}(\mathfrak{p}_i) \text{ for all } i. \end{array} \right.$$

Now we prove that $\text{ht}(\mathfrak{p}_1) \geq n$. To prove this by contradiction, suppose that $l := \text{ht}(\mathfrak{p}_1) \leq n-1$. By $(*)$ we see that $A_{\mathfrak{p}_1}$ is regular. Hence replacing A by A_x for some $x \in A - \mathfrak{p}_1$, we may assume that

$$\left\{ \begin{array}{l} c_1, \dots, c_l \text{ is an } A\text{-sequence in } \mathfrak{p}_1 \text{ (with some } c_i \in \mathfrak{p}_1), \\ (c_1, \dots, c_l) = \mathfrak{p}_1, \\ I = \mathfrak{p}_1 \text{ (hence } \overline{R_n(A)^c} = V(\mathfrak{p}_1)), \\ (R_n) \text{ holds for } A/\mathfrak{p}_1 \cdots (**). \end{array} \right.$$

Since $R_n(A)^c \neq \emptyset$, one can take $\mathfrak{p} \in R_n(A)^c$. Then we have $\mathfrak{p}_1 \subseteq \mathfrak{p}$. To show that $A_{\mathfrak{p}}$ satisfies (R'_n) , we take $\mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and that $\text{codim}(\mathfrak{p}'A_{\mathfrak{p}}) \leq n$. There exists $\mathfrak{p}'' \in V(\mathfrak{p}' + \mathfrak{p}_1)$ such that $\text{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_{\mathfrak{p}}) = \text{codim}((\mathfrak{p}''/\mathfrak{p}_1)A_{\mathfrak{p}}) (= \text{codim}(\mathfrak{p}''A_{\mathfrak{p}}/\mathfrak{p}_1A_{\mathfrak{p}}))$. We

have

$$\begin{cases} \text{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_{\mathfrak{p}}) = \text{ht}(\mathfrak{p}/\mathfrak{p}_1) - \text{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1) \\ \quad = \text{ht}(\mathfrak{p}) - l - \text{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1), \\ \text{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_{\mathfrak{p}}) = \text{codim}((c_1, \dots, c_l)(A/\mathfrak{p}')_{\mathfrak{p}/\mathfrak{p}'} \leq l, \\ \text{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_{\mathfrak{p}}) = \text{ht}(\mathfrak{p}/\mathfrak{p}') - \text{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1). \end{cases}$$

It follows that

$$\begin{aligned} \text{codim}(\mathfrak{p}''A_{\mathfrak{p}}/\mathfrak{p}_1A_{\mathfrak{p}}) &= \text{codim}((\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_{\mathfrak{p}}) \\ &\leq \text{ht}(\mathfrak{p}) - \text{ht}(\mathfrak{p}/\mathfrak{p}') \\ &= \text{codim}(\mathfrak{p}'A_{\mathfrak{p}}) \leq n. \end{aligned}$$

By (**) we see that (R_n) holds for $(A/\mathfrak{p}_1)_{\mathfrak{p}/\mathfrak{p}_1} = A_{\mathfrak{p}}/\mathfrak{p}_1A_{\mathfrak{p}}$. By Lemma 3.1, we see that $A_{\mathfrak{p}''}/\mathfrak{p}_1A_{\mathfrak{p}''} = (A_{\mathfrak{p}}/\mathfrak{p}_1A_{\mathfrak{p}})_{\mathfrak{p}''A_{\mathfrak{p}}/\mathfrak{p}_1A_{\mathfrak{p}}}$ is regular, which shows that $A_{\mathfrak{p}''}$ is regular. It follows that $(A_{\mathfrak{p}})_{\mathfrak{p}'A_{\mathfrak{p}}} = (A_{\mathfrak{p}'})_{\mathfrak{p}'A_{\mathfrak{p}'}}$ is regular. Therefore we see that $A_{\mathfrak{p}}$ satisfies (R'_n) . Let $\mathfrak{q} \in \text{Spec}(A)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. If $\mathfrak{q} \in R_n(A)$, then (R_n) holds for $A_{\mathfrak{q}}$, hence (R'_n) holds for $A_{\mathfrak{q}}$. If $\mathfrak{q} \in R_n(A)^c$, then we see that (R'_n) holds for $A_{\mathfrak{q}}$, discussing in the same way as above. Thus, it follows from Lemma 3.1 that (R_n) holds for $A_{\mathfrak{p}}$. Since $\mathfrak{p} \in R_n(A)^c$, we have a contradiction. Thus we have shown that $\text{ht}(\mathfrak{p}_1) \geq n$, hence $\text{ht}(I) \geq n$.

Therefore we can arrange the order of $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ to satisfy the following conditions.

$$\text{ht}(\mathfrak{p}_i) \begin{cases} = n & (1 \leq i \leq s), \\ > n & (s < i \leq t), \end{cases}$$

$$A_{\mathfrak{p}_i} \text{ is } \begin{cases} \text{non-regular} & (1 \leq i \leq r), \\ \text{regular} & (r < i \leq s). \end{cases}$$

Put $J = \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_r$. Let $\mathfrak{p} \in R_n(A)^c$. Then there exists $\mathfrak{p}' \in \text{Spec}(A)$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$, $\text{ht}(\mathfrak{p}') \leq n$, and that $A_{\mathfrak{p}'}$ is non-regular. By (*) we get $\text{ht}(\mathfrak{p}') = n$. Replacing $\mathfrak{p}$ by $\mathfrak{p}'$, we may assume that $\text{ht}(\mathfrak{p}) = n$. Since $R_n(A)^c \subseteq V(I)$, we have $I \subseteq \mathfrak{p}$, hence $\mathfrak{p}_k \subseteq \mathfrak{p}$ for some k . Since $\text{ht}(\mathfrak{p}) = n$, we have $\mathfrak{p}_k = \mathfrak{p}$ and $1 \leq k \leq s$, and since $A_{\mathfrak{p}}$ is non-regular, we have $1 \leq k \leq r$. It follows that $J \subseteq \mathfrak{p}_k = \mathfrak{p}$, i.e. $\mathfrak{p} \in V(J)$. Therefore, we have $R_n(A)^c \subseteq V(J)$. Since the opposite inclusion is obvious by the choice of $\mathfrak{p}_i$, we have $R_n(A)^c = V(J)$. Thus, we get $R_n(A) = D(J)$, which shows that $R_n(A)$ is open in $\text{Spec}(A)$. $\square$

Added in Proof: The author was informed that the results similar to the present paper had been reported in the following paper :
C.Massaza e P.Valabrega, Sull'apertura di luoghi in uno schema localmente noetheriano, Boll. U.M.I. 14 (1977),564-574.

REFERENCES

- [1] W. BRUNS and J. HERZOG, *Cohen-Macaulay rings*, Cambridge Univ. Press, 1993.
- [2] S. GRECO and M. MARINARI, Nagata's criterion and openness of loci for Gorenstein and complete intersections, *Math. Z.* **160** (1978), 207-16.
- [3] H. MATSUMURA, *Commutative algebra, second edition*, Benjamin, 1980.
- [4] H. MATSUMURA, *Commutative ring theory*, Cambridge Univ. Press, 1980.
- [5] M. NAGATA, *Local rings*, Interscience, 1962
- [6] M. NAGATA, On the closedness of singular loci, *Publ. Math. IHES* **2** (1959), 29-36.

RYO TAKAHASHI

GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY

OKAYAMA UNIVERSITY

OKAYAMA 700-8530, JAPAN

(*Received April 6, 2000*)