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Introduction. In the paper a metric space R means a G-space[1] such that for two distinct points there exists a unique geodesic which passes through these points and any three non-colinear points lie on a 2-flat, i. e., which is said to be Desarguesian in H. Busemann's sense. Let $\mathfrak{l}$ be a geodesic and p and q two distinct points on $\mathfrak{l}$. If there exists a motion Φ which carries p to q and $\mathfrak{l}$ into itself, Φ is said a translation along $\mathfrak{l}$. We prove that, if for any geodesic there exists a transitive group of translations along this geodesic, the space is Minkowskian, hyperbolic or elliptic.

If R is of 2-dimensions in Menger-Uryson's sense and Desarguesian, the space is a manifold and the above holds [1][2]. The main purpose of the paper is to prove the above generally in finite dimensional case not less than 3. In §**1** we show that, if in Hilbert geometry a space admits translations such as in the above, the space is hyperbolic. By use of this the above is proved in §2. In §**3** we give some remarks concerned to the results of the previous paper [4].

§ **1.** Let A^n be an n-dimensional affine space ($n \geqq 3$). A point with coordinates $(x^1, \cdots, x^n)$ is denoted by x. Let K be a convex body with interior points and $\mathfrak{L}$ an affine line in A^n. We assume that $\mathfrak{L}$ intersects K at two distinct points c and d. Then we define the distance $\rho(a_1, a_2)$ between two points a_1 and a_2 on the affine segment cd to be

$$(1.1) \qquad \rho(a_1, a_2) = \mathrm{k}\left|\log\frac{1-\tau_1}{1-\tau_2}\Big/\frac{\tau_1}{\tau_2}\right| (\mathrm{k}>0),$$

where $a_1^i = (1-\tau_1)c^i + \tau_1 d^i$ and $a_2^i = (1-\tau_2)c^i + \tau_2 d^i (i = 1, 2, \cdots, n)$. The metric space R thus defined in the interior K^0 of K is said an n-dimensional H-space and the metric is said Hilbertian. In R the affine segment a_1a_2 is

1) A space R is said to be a G-space if the following axioms are fulfilled.

A. R is metric with distance xy $(= yx)$.

B. R is finitely compact, i. e., a bounded infinite set has at least one accumulation point in R.

C. Given two distinct points x, z then a point y with $(x\ y\ z)$, i. e., different from x and z with $xy+yz=xz$ exists.

D. To every point p of R there corresponds a positive number ρ_p such that for any two distinct points x, y in $S(p, \rho_p)$ (i. e., xp, $yp<\rho_p$) a point z with $(x\ y\ z)$ exists.

E. If $(x\ y\ z_1)$, $(x\ y\ z_2)$ and $yz_1=yz_2$ then $z_1=z_2$.

2) Numbers in brackets refer to the references at the end of the paper.

a shortest connection from a_1 to a_2 (or from a_2 to a_1). If K does not contain a coplaner segment, a_1a_2 is a unique shortest connection.

Let $\mathfrak{l}$ be the open affine segment cut off by K^0 from the affine line $\mathfrak{L}$. Then $\mathfrak{l}$ is said a straight line. Next we prove the following

(1. 2) Theorem. *Let* R *be an n-dimensional H-space. If for any straight liue* $\mathfrak{l}$ *the space admits a transitive group of translations along* $\mathfrak{l}$, *then* R *is hyperbolic.*

We prove firstly Propositions (1. 3) and (1. 4) and the theorem is proved lastly.

(1. 3) *Under the same assumption as in Theorem* (1. 2), *the convex surface* K *is differentiable.*

Proof. Since K is ccnvex, K is almost everywhere differentiable. Let p and q be two distinct differentiable points on K and P and Q the supporting planes of K at p and q respectively. We introduce projective coordinates $(\xi^0, \cdots, \xi^n)$ in such a way that the hyperplanes P and Q are represented as $\xi^0=0$ and $\xi^n=0$ respectively. We assume further that the coordinates of the points p and q are represented as $(0, \cdots, 0, 1)$ and $(1, 0, \cdots, 0)$ respectively. For the sake of brevity we assume Q has only only one point q with K.

Let $\mathfrak{l}$ be the straight line cut off by K^0 from the affine line through p and q and Φ a translation along $\mathfrak{l}$. Let a and b be two points on $\mathfrak{l}$ and suppose that the translation Φ carries a to b. Further let A and B be the hyperplanes determined by a, $P\cap Q$ and b, $P\cap Q$ respectively. Then Φ carries A to B, i. e., $A\Phi=B$. If we put

$$a=p+\sigma q \text{ and } b=p+\sigma' q,$$

the trnslation Φ is represented as $\sigma'=k\sigma$ where k is a positive constant.

Let A_1 and A_2 be the two parts of K divided by A, i. e., we assume that $A_1\cap A_2=\phi$, $A_1\cup A_2\cup(K\cap A)=K$ and $A_1\ni p$, $A_2\ni q$. Simiiarly let B_1 and B_2 be two parts divided by B such that $B_1\cup B_2\cup(K\cap B)=K$ and $B_1\ni p$, $B_2\ni q$. Then we have $A_i\Phi=B_i$ $(i=1, 2)$. This is clear from $P\Phi=P$, $Q\Phi=Q$ and $A\Phi=B$.

In the original affine space A^n the system of all affine lines through the point p and points of $A\cap K$ forms a cone Z and similarly the system of all affine lines through p and points of $B\cap K$ forms a cone Z'. For convenience's sake we assume $\sigma<\sigma'$. Then the interior of Z contains that of Z'. Let c be a point of R which is not on $\mathfrak{l}$ and $\mathfrak{x}$ the asymptote to $\mathfrak{l}$ through c. Let C be the hyperplane determined by the point c and $P\cap Q$ and c' the point at which C intersects $\mathfrak{l}$. Consider the sequence of straight lines $\{\mathfrak{x}\Phi^n\}$ $(n=1, 2, \cdots)$. Let c_n be the point at which each $\mathfrak{x}\Phi^n$ intersects C. Then $C\supset\{c_n\}$. Next we prove that the sequence of points $\{c_n\}$ converges to the point c' in the sense of the metric ρ, i. e., in the sense of the original affine topology.

Let x be a point of $\mathfrak{x}$, and y the point at which the hyperpiane determined by x and $P\cap Q$ intersects $\mathfrak{l}$. Then $\rho(x, y)\to 0$ as x tends to p. This is clear from that K is differentiable at p. It follows from this that the sequence of points $\{c_n\}$ converges to the point c'.

Let d be the point distinct from p at which the affine line containing $\mathfrak{x}$ intersects K. Then the sequence $\{d\Phi^n\}$ converges to q in the sense of the affine topology. To prove the proposition, suppose that K is not differentiable at d. Then there exists distinct supporting planes D_1 and D_2 at d. Let r be the point at which the affine line through p and d intersects hyperplane Q. Further let p_1 and p_2 be the points at which D_1 and D_2 intersect the affine line thrugoh q and r and s the point at which the affine line through q and r intersects $P\cap Q$. Then the order of the points r, s, p_1 and p_2 is supposed to be srp_1p_2. Hence the double ratio $(sr,\ p_1p_2)$ is positive. By choosing D_1 and D_2 suitably, if necessary, we assume $(sr,\ p_1p_2)\neq 1$.

On the other hand the sequence of points $\{r\Phi^n\}$ converges to the point q. Let $\mathfrak{L}$ be the affine line through q and r. Then there exists a subsequence $\{\mathfrak{L}\Phi^{n_i}\}$ of $\{\mathfrak{L}\Phi^n\}$ which converges to an affine line through q lying on the hyperplane Q. The sequence of points $\{p_1\Phi^{n_i}\}$ and $\{p_2\Phi^{n_i}\}$ converges to q and the seqnence of hyperplanes $\{D_1\Phi^{n_i}\}$ and $\{D_2\Phi^{n_i}\}$ converge to Q. It is easy to see from this that the sequence of positive numbers $\{(r\Phi^{n_i}s\Phi^{n_i},\ p_1\Phi^{n_i}p_2\Phi^{n_i})\}$ converges to 1. But each $(r\Phi^{n_i}s\Phi^{n_i},\ p_1\Phi^{n_i}p_2\Phi^{n_i})$ equals $(rs,\ p_1p_2)$, since each Φ^{n_i} is represented as a projective transformation. But this is a contradiction.

Even if Q has common points other than q with the surface K, we arrive at the same conclusion as in the above by slight modifications. Thus we end the proof.

(1. 4) *Under the same assumption the surface* K *is strictly convex.*

Proof. We use the same notations as before. Suppose indeed that the supporting planes Q has common points other than q with K. Then K has a common segment with an affine line through q. At least one of the end points of this segment does not coincide with q. We denote by q' such an end point. Let $\mathfrak{y}$ be the intersection of the affine line through p and q' with K^0. Then $\mathfrak{y}$ is a straight line of R and an asymptote to $\mathfrak{l}$. Let y be a point on $\mathfrak{y}$ and x a point at which the hyperplane X through $P\cap Q$ and y intersects $\mathfrak{l}$. If y tends to q', then $\rho(x, y)\to\infty$ and if y tends to p, then $\rho(x, y)\to 0$.

Let y_0 be a fixed point on $\mathfrak{y}$ and x_0 the point of the intersection of the hyperplane X_o through $P\cap Q$ and y_0 with the straight line $\mathfrak{l}$. Suppose further that $\sigma<\rho'$ $(a\Phi=b,\ a=p+\sigma q,\ b=p+\sigma' q)$. Then the point q' is not necessarily invariant under Φ but the set of points $\{q'\Phi^n\}$ $(n=0, \pm 1, \pm 2, \cdots)$ lies on Q. Obviously each of straight lines $\mathfrak{y}\Phi^{-n}$ $(n=0, 1, 2, \cdots)$ intersects X_0 at a point c_n. Then the sequence of points $\{c_n\}$ is bounded on X_0.

On the other hand, if y tends to q', then $\rho(x, y) \to \infty$. Hence the sequence $\{c_n\}$ is not bounded. This contradicts the above. The proposition follows from this.

Now we prove Theorem (1. 2).

Proof. Let A and B be hyperplanes which intersect K and such that $(\overline{\mathrm{K}}^0 \cap A) \cap (\overline{\mathrm{K}}^0 \cap B) = \phi$. Then $A \cap B$ is disjoint from $\overline{\mathrm{K}}^0$. Let P and Q be the supporting planes of K at p and q respectively. Let $\mathfrak{l}$ be the straight line of R which coincides with the intersection of the affine line through p and q with $\overline{\mathrm{K}}^0$. Let a and b be the points at which $\mathfrak{l}$ intersects A and B respectively. Then $\mathfrak{l}$ is perpendicular to A and B at a and b respectively. There exists a translation Φ_1 which carries a to b.

Let C be a hyperplane which intersects K and C_1 and C_2 be two parts of K divided by C such that $C_1 \cap C_2 = \phi$ and $C_1 \cup C_2 \cup (\mathrm{K} \cap C) = \mathrm{K}$. Suppose further that one of C_1 and C_2 contains $\mathrm{K} \cap \mathrm{A}$ and $\mathrm{K} \cap B$. Since $B \cap C$ is disjoint from K, there exist the supporting planes S and T through $B \cap C$. We denote by s and t the common points of K with S and T respectively. Let $\mathfrak{x}$ be the straight line which coincides with the intersection of K^0 with the affine line through s and t. Let b' and c be the points at whicht $\mathfrak{x}$ intersects B and C respectively. Then there exists a translation Φ_2 along $\mathfrak{x}$ such that $b'\Phi_2 = c$. Obviously $\overline{\mathrm{K}}^0 \cap B$ is carried onto $\overline{\mathrm{K}}^0 \cap C$ under Φ_2 and $\overline{\mathrm{K}}^0 \cap A$ on to $\overline{\mathrm{K}}^0 \cap B$ under Φ_1.

Under the same consideration as in the above for the hyperplanes A and C there exists a translation Φ_3 along the straight line which is the intersection of $\overline{\mathrm{K}}^0$ with the affine line through the tangent points of the supporting planes through $A \cap C$. If we put $\Psi = \Phi_3\Phi_2\Phi_1$, Ψ is a motion of R and $\overline{\mathrm{K}}^0 \cap A$ is carried into itself under Ψ. Let A_1 and A_2 be two parts of K divided by A such that $A_1 \cap A_2 = \phi$ and $A_1 \cup A_2 \cup (\mathrm{K} \cap A) = \mathrm{K}$. It is easy to see that the motion Ψ carries A_1 into A_2 and A_2 into A_1.

On the other hand, since the motion Ψ is represented as a projective transformation which carries K into itself, the motion Ψ can be represented as

$$x'_i = \frac{\sum_{k=1}^{n} \mathrm{a}_{ik}\mathrm{x}^k + \mathrm{a}_i}{\sum_{k=1}^{n} \mathrm{a}_{n+1k}\mathrm{x}^k + \mathrm{a}_{n+1}} \qquad (i = 1, 2, \cdots, n)$$

where $\sum_{k=1}^{n} \mathrm{a}_{n+1k}\mathrm{x}^k + \mathrm{a}_{n+1} \neq 0$ for $x \in \overline{\mathrm{K}}_0 \cap A$. Hence Ψ is a continuous mapping of $\overline{\mathrm{K}}^0 \cap A$ and by Brouwer's fixed point theorem there exists a point fixed under Ψ. We denote by r such a fixed point. Then we can consider the following two cases:

i) The point r is a boundary point of $\overline{\mathrm{K}} \cap A$.

ii) The point r is an inteiror point of $\overline{\mathrm{K}}^0 \cap A$.

Suppose that the case i) holds. Let $\{r_n\}$ be a sequence of points on $\mathrm{K}^0 \cap A$ which converges to r in the sense of the original affine topology and $\mathfrak{L}_n$ the affine line which is the carrier of the straight line perpendicular to $\mathrm{K}^0 \cap A$ at each r_n. Then the sequence of affine lines $\{\mathfrak{L}_n\}$ converges to an affine line $\mathfrak{A}$ tangent to K at r. If we replace the hyperplane at infinity by the hyperplane A' tangent to K at r, then Ψ is represented as an affine transformation, since $\mathfrak{A}$ lies on A' and under Ψ A' and $\mathfrak{A}$ are carried into themselves respectively.

Since r is invariant under Ψ, it is easy to show that there exists on $\mathfrak{A}$ a point u fixed under Ψ distinct from r. A hyperplane D through $A \cap A'$ is carried into a hyperplane D' throngh $A \cap A'$. Let an affine line $\mathfrak{B}$ through u intersects D and D' at points d and d' respectively. Then the affine center of d and d' coincides with the intersection f of the affine line $\mathfrak{B}$ with the hyperplane A. We show that the tangent cone of K whose vertx is u is tangent to K at points of $A \cap \mathrm{K}$. To do this let $\mathfrak{B}$ intersect K at points e and e'. Then we show that the four points u, e, e' and f is in the order $uefe'$ or $ue'fe$.

Let E and E' be the hyperplanes through e, $A \cap A'$ and e', $A \cap A'$ respectively and suppose that the order of the points u, e, e' and f is $uee'f$. Then $(uf, ee') = (uf\Psi, e\Psi e'\Psi)$. Let the affine line $\mathfrak{B}$ intersect $E\Psi$ and $E'\Psi$ at points $\bar{e}$ and $\bar{e}'$. Then the affine center of e and $\bar{e}$ coincides with f and similarly the affine center of e' aod $\bar{e}'$ is also identical with f. It is easy to see

$$(uf,\ ee') = (uf\Psi, e\Psi\ e'\Psi) = (uf,\ \bar{e}\,\bar{e}').$$

But this is a contradiction. For if we put $e = (1-\tau)f + \tau u$ and $e' = (1-\tau')f + \tau' u$, the points $\bar{e}$ and $\bar{e}'$ are represented as $(1+\tau)f - \tau u$ and $(1+\tau')f - \tau' u$. We have then

$$(uf,\ e\,e') = \frac{1-\tau}{1-\tau'} \Big/ \frac{\tau}{\tau'} \quad \text{and}$$

$$(uf,\ \bar{e}\,\bar{e}') = \frac{1+\tau}{1+\tau'} \Big/ \frac{\tau}{\tau'}.$$

Hence $(uf,\ ee') \neq (uf,\ \bar{e}\,\bar{e}')$ unless $\tau = \tau'$. Thus we see $uefe'$ or $ue'fe$. It is easy to see from this that a tangent line of K through u is tangent to K at a point of $\mathrm{K} \cap A$. Our assertion follows from this. The same fact holds in the case ii). Next we show this.

The straight line $\mathfrak{a}$ through r, which is perpendicular to A at r, is carried into itself under Ψ. Let p and q be the points at which the affine line $\mathfrak{A}$ containing $\mathfrak{a}$ intersects K and P and Q the supporting planes of K at p and q respectively. Then $p\Psi = q$ and $q\Psi = p$ and hence $P\Psi = Q$ and $Q\Psi = P$. It follows from this that $P \cap Q$ is contained in the hyperplane

A. Let u be the point on the affine line $\mathfrak{A}$ such that $(pq, ru) = -1$. Then we have $u\Psi = u$. If we replace the hyperplane at infinity by the hyperplane through $P \cap Q$ and the point u, Ψ is represented as an affine transformation. It is easy to see that the tangent cone whose vertex is u is tangent to K at points of $A \cap K$. The arguments are quite parallel to the above. Thus the theorem is proved.

§ 2. Let R be an n-dimensional G-space ($n \geqq 3$) and suppose that for two distinct points there eists a unique geodesic through these points. If for any three non-collinear points of R there exists a subset E containing these points which is a 2-dimensional G-space under the metric of R, the space is Desarguesian[3]. Such a subset E is said a 2-flat.

If R is an n-dimensional G-space such as in the above, the space has one of the following three properties:

I. **R** is straight and topologically mapped onto the interior of a convex subset K of an n-dimensional affine space A^n in such a way that every geodesic of **R** is mapped onto the intersection of K with an affine line.

II. The geodesic are great eircles and have same length. The space is topologically mapped onto an n-dimensional projective space in such a way that every geodesic of **R** is mapped onto a projecfive line.

III. **R** is straight and topologically mapped onto an n-dimensional affine space A^n in such a way that every geodesic of R is mapped onto an affine line.

The space is said to be of *I-*, *II-* or *III-type* according as the space has the above properties, *I*, *II* or *III*. Next we prove the following

(2. 1) *Let* R *be an n-dimensional G-space of I-type* ($n \geqq 3$). *If for any geodesic the space* R *admits a transitive group of translations along this geodesic, then the space in an n-dimensional H-space.*

Proff. Suppose that the space is the interior of a convex set K with interior points. Under the above assumption there exist two distinct supporting planes P and Q. Let p and q be their touch points respectively. We use the same notations as before and introduce by adding the infinite plane to A^n a system of projective coordinates $(\xi^0, \xi^1, \cdots, \xi^n)$ as in § **1**. Let $\mathfrak{l}$ be the intersection of $\mathfrak{L}$ with the convex subset K and Φ be a translation along $\mathfrak{l}$ such that $a\Phi = b$ where $a = p + \sigma q$ and $b = p + \sigma' q$ $(\sigma < \sigma')$. Then

3) If a G-space satisfies the following conditions, we say that the space has Desarguesian property or the space is Desarguesian.

(1) If the geodesics $g(a_1, a_2)$, $g(b_1, b_2)$ and $g(c_1, c_2)$ have a common point and the intersections p $(= g(a_1, b_1) \cap g(a_2, b_2))$ and p $(= g(b_1, c_1) \cap g(b_2, c_2))$ exist, then, if two of the three intersections exist, they coincide.

(2) If the intersections $g(a_1, b_1) \cap g(a_1, b_2)$, $g(b_1, c_1) \cap g(b_2, c_2)$ and $g(c_1, a_1) \cap g(c_2, a_2)$ exist and are collinear and if two of the three intersections $g(a_1, a_2) \cap g(b_1, b_2)$, $g(b, b_2) \cap g(c_1, c_2)$ and $g(c_1, c_2) \cap g(a_1, a_2)$ exist, they coincide.

on l the motion Φ is represented as $\sigma' = k\sigma (k > 0)$. If we put

$$\rho(a, b) = f(\sigma, \sigma') = -f(\sigma', \sigma),$$

we have $f(\tau\sigma, \tau\sigma') = f(\sigma, \sigma)$ $(\tau > 0)$ and further we have

$$f(\sigma, \sigma') + f(\sigma', \sigma'') = f(\sigma, \sigma'') \quad \text{for } \sigma'' > \sigma' > \sigma.$$

If we put $\tau = \sigma'/\sigma$, we have from the above

$$\frac{f(\sigma, \sigma')}{\sigma' - \sigma} = \frac{f(1, \tau)}{\sigma(\tau - 1)}.$$

On the other hand, the function $f(1, \tau)$ is an increasing function of τ. Hence it is easily seen that $\lim_{\tau \to 1} f(1, \tau)/(\tau - 1)$ exists. If we denote by k_0 this limit, we have

$$\lim_{\sigma' \to \sigma} \frac{f(\sigma, \sigma')}{\sigma' - \sigma} = \frac{k_0}{\sigma} \text{ for any } \sigma > 0,$$

i. e., the function $f(\sigma, \sigma')$ is differentiable. It follows from this that the space **R** is an H-space, since K does not contain a half affine line. The proposition follows from this.

Thus we see that, if the space **R** is of *I-type* and if for any geodesic **R** admits a transitive group of translations along this geodesic, then **R** is hyperbolic. It is also easy to see that, if the space **R** is of *II-type* and admits a group of translations such as in the above, then **R** is elliptic.

Now we consider the case where the space is of *III-type*. Then **R** is straight and every geodesic is an affine line. A translation Φ along a straight line carries an affine line into an affine line and is a one-one mapping of the affine space A^n into itself. Hence Φ is represented as an affine transformation. Thus we see that for two distinct points a and b the affine center of these points coincides with the mid-point of these points in the sense of the metric ρ. It follows from this that spheres are convex. Since the space satisfies the parallel axiom, the space is Minkowskian [1]. Thus we have the following

(2. 2) Theorem. *Let* **R** *be an n-dimensional G-space* $(n \geqq 3)$ *and suppose that for any two distinct points there exists a unique geodesic through these points and any three non-collinear points lie on a 2-flat. If for any geodesic* **R** *admits a transitive group of translations along this geodesic, the space is Minkowskian, hyperbolic or elliptic.*

If **R** is locally symmetric (or globally symmetric), then for any geodesic **R** admits also a transitive group of translations along this geodesic. Hence if R is a 2-dimensional Desarguesian space or an n-dimensional G-space $(n \geqq 3)$ such that any three non-collinear points lie on a 2-flat and locally symmetric (or globally symmetric), the space is Minkowskian, hyperbolic or elliptic.

§ 3. Let p be a point of $\mathbf{R}$ and $\mathfrak{F}$ the system of all straight lines through p. Then $\mathbf{R}$ is an n-dimensional H-space and admits a tiansitive group of motions Γ such that a straight line of $\mathfrak{F}$ is carried into a straight line of $\mathfrak{F}$ under an element of Γ, the space is hyperbolic [4]. The following is also clear.

(3. 1) Theorem. *Let* R *be an n-dimensionsional G-space of I-type* ($n \geqq 3$) *and* $\mathfrak{F}$ *the system of all straight lines through a given point p. If* $\mathbf{R}$ *admits a transitive group of motions* Γ *such that a straight line of* $\mathfrak{F}$ *is carried into a straight line of* $\mathfrak{F}$ *under an element of* Γ, *then the space is hyperbolic.*

The theorem holds also for a 2-dimensional Desarguesian space which corresponds to an n-dimensional G-space of I-type ($n \geqq 3$). When the space R is of *II*- or *III-type*, even if the space admits such a group of motions, the space is not necessarily elliptic or Euclidean. If $\mathbf{R}$ is Minkowskian and admits such a group of motions, then the space is Euclidean. Hence a space of *III-type* is not necessarily Minkowskian. We can explain these circumstances by showing some examples. For example the space constructed in [2] (See p. 61) by H. Busemann permits rotations about the origin but is not Euclidean. It is also easy to construct a space of *II-type* which permits rotations about a given point but is not elliptic.

REFERENCES

[1] H. BUSEMANN: The geometry of geodesics, Acad. Press INC., New York (1955).

[2] H. BUSEMANN: Similarities and differentiability, Tohoku Math. J., 9 (1957), 56—67.

[3] P. J. KELLY and L. J. PAIGE: Symmetric perpnedicularities in Hilbert geometries, Pac, J. Math., II (1952), 319—322.

[4] Y. NASU: On Hilbert geometry, Math, J. Okyama Univ., 10 (1961), 101—112.

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