Supplementary remarks to the previous papers

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SUPPLEMENTARY REMARKS TO THE PREVIOUS PAPERS

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The present note contains several improvements of the results obtained in [2], [4] and [5], which are closely related with the recent ones cited in [1]. However, our proofs will be given without making use of Inatomi’s [1].

As to notations and terminologies used in this note, we follow the previous ones [4] and [5]. Now, we shall prove our first lemma.

Lemma 1. Let $U \ni 1$ be an algebra over an infinite field $\phi$ of finite rank, and $T$ an intermediate ring of $U/\phi$. If $U = T[x]$ then $U = T[u]$ with some regular element $u$.

Proof. $A = \phi[x]$ is evidently a commutative subalgebra of $U$. If we denote by $N$ the radical of $A$, then $\overline{A} = A/N = \phi[\overline{x}] = \overline{A}_1 \oplus \cdots \oplus \overline{A}_s$, where $\overline{A}_i$'s are fields over $\phi$ and $\overline{x}$ is the residue class of $x$ modulo $N$. We set here $\overline{x} = \overline{a}_1 + \cdots + \overline{a}_s (\overline{a}_i \in \overline{A}_i)$. Since $\phi$ is infinite, we can find such an element $k \in \phi$ that each $\overline{A}_i$-component of $\overline{x} - k$ is non-zero. And then, it will be clear that $u = x - k$ is a regular element and $U = T[u]$.

In virtue of Lemma 1, we can prove the following sharpening of [4, Theorem 1].

Theorem 1. Let $R$ be a separable simpe algebra over a field $\phi (\subseteq C)$ of finite rank. If $a$ is an arbitrary element of $R \setminus C$ then $R = \phi[a, r]$ for some regular element $r$.

Proof. In any rate, by [4, Theorem 1], $R = \phi[a, b]$ for some $b$. Then, if $\phi$ is infinite, our assertion is clear by Lemma 1. Accordingly, in what follows, we may assume that $\phi$ is finite and $n > 1$, whence $R = (C)_n$. To be easily seen, in the proof (i) of [4, Lemma 10] we may replace the nilpotent element $u^* = \sum e_{i1} e_{i1}$ by the regular element $1 - u^*$. And so, by the proof Case III of [4, Theorem 1] and the proof Case I of [4, Proposition], we obtain eventually $R = \phi[a, r]$ with some regular element $r$.

Now, [1, Theorem 2] is only an easy consequence of Theorem 1, and conversely.

Corollary 1 (Inatomi). Let $R$ be a simple algebra over $C$ of finite
Lemma 2. Let $U \ni 1$ be a ring. A subring of $U$ containing 1 that is represented as $\sum_{i=1}^n A'c_{ij}$ with matrix units $c_{ij}$'s ($m > 1$) and $A' = V_{A}(\{c_{ij}\})$, and $B$ a subring of $U$ such that $B \subseteq B$. 

(i) If $Bc_{pq} \subseteq B$ for some $c_{pq}$ then $BA = B$.

(ii) Let $2A = 0$, and $x$ an element of $A'$ neither 0 nor 1. If $B(x + c_{12}) \subseteq B$ then $BA = B$.

(iii) If $B(\sum_{i=1}^m c_{1i} + c_{1m} + c_{2n}) \subseteq B$ then $BA = B$. ( Needless to say, in case $B$ contains 1, our conclusion $BA = B$ in (i)-(iii) may be replaced by $A \subseteq B$.)

Proof. (i) If $p \neq q$ then $B \supseteq (1 + c_{pq}) Bc_{pq} (1 + c_{pq})^{-1} = B(1 + c_{pq}) c_{pq} (1 - c_{pq}) = B(c_{pq} + c_{qq} - c_{pp} - c_{qq})$ and $B \supseteq Bc_{pq}$ imply $B \supseteq B(c_{pq} - c_{pp} - c_{qq})$, whence it follows $B \supseteq Bc_{pq} (c_{pq} - c_{pp} - c_{qq}) = B(c_{pq} - c_{pp})$. Again by our assumption, we obtain $B \supseteq Bc_{pp}$. Now, for each $i \neq j$ and $a' \in A'$, $B \supseteq (1 + a'c_{ij}) Bc_{pp}$, $(1 + a'c_{ij})^{-1} = B(1 + a'c_{ij}) Bc_{pp} (1 - a'c_{ij}) = B(c_{pp} + a'c_{ij})$ and similarly $B \supseteq (1 + a'c_{ij}) Bc_{pp} (1 + a'c_{ij})^{-1}$, $B(c_{pp} + a'c_{ij})$. Accordingly, we have $B \supseteq Ba'c_{ij}$ and $Ba'c_{ij}$. From those, it will be easy to see that $B \supseteq BA$.

(ii) We set $a_i = x + c_{12}$ and $a^*_i = c_{pq} + c_{1m} + c_{2n}$. Then, $B \supseteq (1 + c_{12})$ $Ba_i$ $(1 + c_{12})^{-1} = B(a_i + a^*_i)$ implies $B \supseteq B a_i^*$. And further, $B \supseteq B(a^*_i - a^*_j) = Bc_{12}$. Now, $BA = B$ is a direct consequence of (i).

(iii) Setting $a_2 = \sum_{i=1}^m c_{1i} + c_{1m} + c_{2n} B \supseteq (1 + c_{12})$ $B a_2 (1 + c_{12})^{-1} = B(a_2 + c_{1i})$ implies $B \supseteq Bc_{12}$. And so, again by (i), we obtain $BA = B$.

Next, we expose our second theorem. It contains [1, Lemma 4] and [2, Theorem 3]. And its proof is simpler than that of Inatomi's we think.

Theorem 2. Let $U$ be a ring containing 1, $A \supseteq 1$ a simple subring of $U$ different from $(GF(2))^n$. If $B$ is a two-sided simple subring of $U$ invariant relative to all inner automorphisms effected by regular elements of $A$: $BA = B$, then either $BA = B$ or $A \subseteq V_B(B)$. 

Proof. In fact, for the case where $A$ is a division ring, our proof proceeds just like in that of [3, Lemma 3.5]. And so, the details may be left to readers. While, in case $A$ is not a division ring, we set $A = \sum_{i=1}^m A'c_{ij}$ with matrix units $c_{ij}$'s ($m > 1$) and a division ring $A' = V_{A}(\{c_{ij}\})$. One may remark here that $V_B(B)$ is also invariant relative to $A$, and that the same argument as in the proof of [3, Lemma 3.5] proves that for each biregular element $a$ of $A$ (i.e. $a$ and $1 - a$ are regular) there holds either $BA \subseteq B$ or $a \in V_B(B)$. Now, we shall complete our proof by
distinguishing three cases:

Case I: \( A \) is not of characteristic 2. Evidently, \( a_0 = 2 + c_{12} \) is biregular. And so, as we noted above, either \( B a_0 \subseteq B \) or \( a_0 \in V_0(B) \), that is, either \( B a_0 \subseteq B \) or \( c_{12} \in V_0(B) \). Recalling here that \( V_0(B) \overset{\sim}{=} V_0(B) \), our assertion is clear by Lemma 2 (i).

Case II: \( A \) is of characteristic 2 and \( A' \neq GF(2) \). As \( a_1 = x + c_{12} (x \neq 0, \ 1 \in A') \) is evidently biregular, it follows either \( B a_1 \subseteq B \) or \( a_1 \in V_0(B) \). And then, Lemma 1 (ii) yields at once our assertion.

Case III: \( A \) is of characteristic 2 and \( m \geq 2 \). In this case, to be easily verified, \( a_2 = \sum c_{i_1} + c_{i_2} + c_{i_3} \) is biregular. And so, this time, our assertion is a consequence of Lemma 1 (iii).

Combining Theorem 2 with [5, Corollary 3.9], one will see at once the next corollary that contains completely [5, Theorem 4.5].

**Corollary 2.** Let a simple ring \( R \) be locally finite and \( h \)-Galois over a simple subring \( S, [R : H]_1 \subseteq S_0 \) and \( T \) an \( f \)-regular intermediate ring of \( R/S \). Then, \( T \) is \( \Theta \)-normal if and only if \( T/S \) is Galois and either \( T \subseteq H \) or \( V \subseteq T \), provided \( V \) is different from \( (GF(2))_p \).

Next, we shall prove the following improvement of [4, Theorem 4] that contains [1, Theorem 6] as well.

**Theorem 3.** Let a simple ring \( R \) be Galois and finite over a simple subring \( S \), and \( T \) a \( \overline{V} \)-normal simple intermediate ring of \( R/S \). Then, \( n(T/S) = 1 \) if and only if \( S \not\subseteq V_r(T) \) or \( T \) is commutative.

**Proof.** It will suffice to prove the if part only. For the case where \( [S : Z] = \infty \), we have seen in [3, Corollary 2.1] that our assertion is true even for arbitrary intermediate ring \( T \). And so, in what follows, we may restrict our attention to the case \( [S : Z] < \infty \) (whence \( [R : C] < \infty \) by [6, Lemma]), and we distinguish two cases:

Case I: \( C \) is finite. Since \( (R \text{ and so}) T \) is finite, \( T/S \cap C \) is a simple algebra of finite rank. If \( S \not\subseteq V_r(T) \), by Theorem 1, \( T = S[T] \) with some regular \( t \). On the other hand, if \( T = V_r(T) \), there is nothing to prove.

Case II: \( C \) is infinite. By Theorem 2, we obtain \( V \subseteq T \) or \( T \subseteq H \). Since, in case \( S \not\supseteq Z \) our assertion is clear by [4, Theorem 3], in what follows, we shall assume that \( S = Z \). Now, \( S = Z \subseteq C_0 = V_r(V) \) implies \( V_r(C_0) = V = V_r(V_r(V)) \), whence it follows \( H \) is commutative: \( H = C_0 \). If \( T \) is commutative, then \( T \subseteq H \) in any case. Accordingly, \( n(T/S) = 1 \) by [4, Theorem 2]. On the other hand, if \( S \not\subseteq V_r(T) \) then \( T \subseteq H \) yields the contradiction \( S \subseteq T = V_r(T) \). It follows therefore \( V \subseteq T \), whence \( V_r(T) \subseteq V_r(T) \subseteq H = C_0 \). Hence, \( R, T \) and \( S \) satisfy the assumptions in [4, Proposition]. And so, if \( s \) is an arbitrary element of \( S \setminus V_r(T) \), then
there exists some $t$ such that $T = Z[s, t] = S[t]$.

We shall present here another proof to [1, Theorem 5].

Corollary 3. Let a simple ring $R$ be Galois and finite over a simple subring $S$. If $T$ is a $V$-normal simple intermediate ring of $R/S$, then $T = S[t, t']$ with some $t$ and $r$.

Proof. By Theorem 3, it will suffice to prove our assertion for the case $S \subseteq V_r(T)$ (whence $[R : C] < \infty$ by [6, Lemma]). If $C$ is finite, the finite ring $T$ is a separable algebra over $S$. And then, our assertion is clear by [7, Theorem 2]. On the other hand, if $C$ is infinite, $T \subseteq H$ or $T \supseteq V( = V_{r}(S) \supseteq V_{r}(V_{r}(T)) = T$, whence $T = V$) by Theorem 2. Hence, our assertion is a consequence of [4, Theorem 2] and [7, Theorem 1].

Finally, we shall prove a partial extension of [4, Theorem 6]. To this end, the next lemma will be needed.

Lemma 3. Let a simple ring $R$ be Galois and finite over a simple subring $S$, and $T$ a regular intermediate ring of $R/S$. If $S \subsetneq V_{r}(T)$ then $n(T/S) \leq \max\{0, n(Z[V_{r}(T)]/Z) - [S : Z]\} / 2$.

Proof. In case $[S : Z] = \infty$, our assertion is clear by [3, Corollary 2.1]. And so, we may, and shall, restrict out proof to the case $[S : Z] < \infty$ (whence $[T : V_{r}(T)] < \infty$ by [6, Lemma]). Then, by Theorem 1, $T = V_{r}(T)[S, u]$ for some $u$. Now, let $S = \sum_{i=1}^{k} Z d_{i}$ and $Z[V_{r}(T)] = Z[a_{1}, \ldots, a_{k}]$ where $h = n(Z[V_{r}(T)]/Z)$. We set here $v = \sum_{i=1}^{k} a_{i} d_{i}$, where $s = \min\{k, h\}$. Then, $T' = S[u, v, \{a_{i} : s < i \leq h\}]$ is a simple subring of $T$ by [4, Lemma 11], for $T'[V_{r}(T)] = T$. Noting that $S[V] = S \times_{V} S$, we see that $\{d_{i}\}'s$ is linearly independent over $V$. And so, for any element $x \in V_{h}(T') (\subseteq V)$. $0 = vx - vx = \sum_{i=1}^{s} (xa_{i} - a_{i}x) d_{i}$ yields at once $xa_{i} = a_{i}x (i = 1, \ldots, s)$. It follows therefore that $V_{h}(T') = V_{h}(S[u, a_{1}, \ldots, a_{s}]) = V_{h}(T)$, that is, $T'$ is a regular subring of $R$. Accordingly, $R$ is Galois and finite over $T'$. Recalling here that $V$ is $\otimes$-normal, $0 = v - v \sigma = \sum_{i=1}^{s} (a_{i} - a_{i} \sigma) d_{i}$, with every $\sigma \in \otimes(R/T')$ implies $a_{i} = a_{i} \sigma (i = 1, \ldots, s)$, whence it follows $a_{i} \in T'(i = 1, \ldots, s)$, where we have proved therefore that $T = T'[V_{r}(T')] = T'$. Our lemma is now a direct consequence of $T' = S[u, v, \{a_{i} : s \leq i \leq h\}]$.

Theorem 4. Let a simple ring $R$ be Galois and finite over a simple subring $S$. If $S \supsetneq Z$ then, for any regular intermediate ring $T$ of $R/S$, $n(T/S) \leq n_{0} = \max n(W/Z)$, where $W$ runs over all the intermediate rings of $V/Z$.

Proof. Firstly $S \supsetneq Z$ yields evidently $S \subsetneq V_{r}(T)$. If $n_{0} = 1$, $V$ is commutative and then $n(T/S) = 1$ by [4, Theorem 2]. And so in what follows, we may assume that $n_{0} > 1$. If $n(Z[V_{r}(T)]/Z) - [S : Z] \leq 0$ then $n(T/S) \leq 2$.
by Lemma 3. While, if \( n(Z[V_r(T)]/[Z]) = [S: Z] > 0 \) then \( n(T/S) \leq n(Z[V_r(T)]/[Z]) + [S: Z] + 2 \leq n(Z[V_r(T)]/[Z]) \leq n_0 \) again by Lemma 3.

Another consequence of Lemma 3 is the next

**Corollary 4.** Let a simple ring \( R \) be Galois and finite over a simple subring \( S \). If \( [S: Z] \geq m_0 = \text{Max} \ n(U/Z) \), where \( U \) runs over all the commutative intermediate rings of \( V/Z \), then \( n(T/S) \leq 2 \) for any regular intermediate ring \( T \) of \( R/S \).

**Proof.** In case \( S \not\subset V_r(T) \), our assertion is clear by Lemma 3. On the other hand, if \( S \subset V_r(T) \) then \( T \subset V \), and \( 1 = [S: Z] \geq m_0 \) means \( m_0 = 1 \). Since \( T = V_r(T)[u, v] \) for some \( u, v \) by Theorem 1, it will be easy to see that \( n(T/S) \leq n(V_r(T)[u]/Z) + 1 = 1 + 1 = 2 \).

**REFERENCES**


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Page 160, line 11. For “(iii) If $B(\sum \n c_{i-1} \div c_{im} \div c_{n}) \subseteq B$ then $BA = B$”
read “(iii) Let $2A = 0$ and $m \geq 2$. If $B(\sum \n c_{i-1} + c_{im} \div c_{n}) \subseteq B$ then $BA = B$”.

Page 160, lines 24—25. For “(iii) Setting…, we obtain $BA = B$” read “We set $a_3 = \sum \n c_{i-1} + c_{im} + c_{im-1}$ and $a^{**} = c_{22} + c_{11} + c_{12}$. Then, $B \supseteq (1 + c_{13})B a_3 (1 + c_{13})^{-1} = B (a_3 \div a^{**})$ implies $B \supseteq Ba^{**}$, and hence $B \supseteq B (a^{**} + (a^{**})^3) = Bc_{12}$. Now, $BA = B$ is a consequence of (i)”.

Page 161, line 10. For “$a_2 = \sum \n c_{i-1} + c_{im} + c_{im-1}$” read “$a_2 = \sum \n c_{i-1} + c_{im} + c_{im-1}$”.  

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