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Abstract

We give a lower bound for the LS category of a formal elliptic space in terms of its rational cohomology.

KEYWORDS: LS category, rationally elliptic space, F_0 -space, formal.

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A LOWER BOUND FOR THE LS CATEGORY OF A FORMAL ELLIPTIC SPACE

Dedicated to the memory of Professor Akie TAMAMURA

YASUSUKE KOTANI AND TOSHIHIRO YAMAGUCHI

ABSTRACT. We give a lower bound for the LS category of a formal elliptic space in terms of its rational cohomology.

1. INTRODUCTION

The Lusternik-Schnirelmann (LS) category, $\text{cat } X$, for a space X is the least integer m such that X can be covered by $m + 1$ open sets, each contractible in X . The rational LS category, $\text{cat}_0(X)$, is the least integer n such that $X \simeq_0 Y$ and $\text{cat } Y = n$. A simply connected CW complex X is called (rationally) elliptic if $\dim H^*(X; \mathbb{Q}) < \infty$ and $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$. An elliptic space X has a positive Euler characteristic, i.e., $\sum_i (-1)^i \dim H^i(X; \mathbb{Q}) > 0$, if and only if it satisfies $\chi_\pi(X) = \sum_i (-1)^i \dim \pi_i(X) \otimes \mathbb{Q} = 0$ [6]. Then it is called often an “ F_0 -space”. For example, a homogeneous space G/H , where G is a compact, connected Lie group and H is a closed subgroup of maximal rank, is an F_0 -space. The rational cup length of a space Z , $\text{cup}_0(Z)$, is the greatest integer n such that the n -product $H^+(Z; \mathbb{Q}) \cdots H^+(Z; \mathbb{Q}) \neq 0$. Also the rational Toomer invariant of Z , $e_0(Z)$, is given by using the Sullivan minimal model [4] $\mathcal{M}(Z) = (\wedge V, d)$ as $\sup\{n \mid \text{there is an element } \alpha \in \wedge^{\geq n} V \text{ such that } [\alpha] \neq 0 \text{ in } H^*(Z; \mathbb{Q})\}$. We remark that

$$(*) \quad \text{cup}_0(Y) \leq e_0(Y) = \text{cat}_0(Y) \leq \text{cat } Y$$

for an elliptic space Y (see [3]).

There is a problem [9]: *If Y is elliptic, then $\dim H^*(Y; \mathbb{Q}) \leq 2^{\text{cat } Y}$?* It is true if $\dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} \leq 1$ [9].

Lemma. *For an F_0 -space X , $\dim H^*(X; \mathbb{Q}) \leq 2^{\text{cup}_0(X)}$.*

Especially, if X has the homotopy type of the r -product of even dimensional spheres, then $\dim H^*(X; \mathbb{Q}) = 2^r = 2^{\text{cup}_0 X} = 2^{\text{cat } X}$.

If an elliptic space Y is formal [4, p.156], roughly speaking, if the Sullivan minimal model [4] is a formal consequence of its rational cohomology, it has the rational homotopy type of the total space of a fibration $X \rightarrow E \rightarrow S$ in which X is an F_0 -space and S is the product of odd dimensional spheres or the one point space (see [2]). Then we have

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Theorem. *If an elliptic space Y is formal, then $\dim H^*(Y; \mathbb{Q}) \leq 2^{\text{cup}_0(Y)}$.*

From the above remark (*), we deduce a partial answer for our problem.

Corollary. *If an elliptic space Y is formal, then $\dim H^*(Y; \mathbb{Q}) \leq 2^{\text{cat } Y}$.*

In the next section, we give the proofs. In the third section, we compare our results with the *total rank conjecture* [4, p.516] of Halperin for certain spaces.

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2. PROOFS

Let X be an F_0 -space. Then there is an isomorphism as algebras

$$H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_p]/(f_1, \dots, f_p)$$

and there is an equation

$$\dim H^*(X; \mathbb{Q}) = |f_1| \cdots |f_p| / |x_1| \cdots |x_p|$$

where $\{|x_i|\}_i$ are all even and $f_1, \dots, f_p$ is a regular sequence [6]. Here $|*|$ is the degree of $*$ as an element in a graded algebra and f_i are homogeneous polynomials with no linear terms.

Proof of Lemma. Suppose that $|x_1| \leq \cdots \leq |x_p|$. Let Φ_i denote the set of all monomials occurring in f_i , i.e., $f_i = \sum_j c_{ij} \sigma_{ij}$ for some $c_{ij} \neq 0 \in \mathbb{Q}$ and $\sigma_{ij} \in \Phi_i$. Since $\dim \mathbb{Q}[x_1, \dots, x_p]/(f_1, \dots, f_p) < \infty$, sets $\Phi_1, \dots, \Phi_p$ must satisfy the “polynomial condition” P.C. [5, p.119] due to Friedlander and Halperin: for each s and for each set of s variables $x_{i_1}, \dots, x_{i_s}$, there are at least s sets $\Phi_{j_1}, \dots, \Phi_{j_s}$ containing a monomial in $\mathbb{Q}[x_{i_1}, \dots, x_{i_s}]$ [5, Theorem 3]. By changing the indexes of Φ_i ’s, we can regard Φ_i as an element of $\mathbb{Q}[x_1, \dots, x_i]$ for any i . Thus we may assume that each f_i contains a term of the form $x_1^{k_{i1}} \cdots x_i^{k_{ii}}$ where $k_{i1}, \dots, k_{ii}$ are non-negative integers with $k_{i1} + \cdots + k_{ii} \geq 2$. Then, for each $1 \leq i \leq p$, we have

$$\begin{aligned} |f_i|/|x_i| &= (k_{i1}|x_1| + \cdots + k_{ii}|x_i|)/|x_i| \\ &\leq k_{i1} + \cdots + k_{ii} \\ &\leq 2^{k_{i1} + \cdots + k_{ii} - 1} \\ &\leq 2^{\deg f_i - 1}, \end{aligned}$$

where $\deg f_i$ is the degree of f_i as a polynomial. Thus

$$\dim H^*(X; \mathbb{Q}) = |f_1| \cdots |f_p| / |x_1| \cdots |x_p| \leq 2^{\deg f_1 + \cdots + \deg f_p - p}.$$

Since the Jacobian $\det(\partial f_i / \partial x_j)$ is the fundamental class of the Poincaré duality algebra [8, Proposition 3], we have $\deg f_1 + \cdots + \deg f_p - p \leq \text{cup}_0(X)$. $\square$

Recall the following Lemma in [9] for the proof of Theorem.

Lemma 2.1 ([9, Lemma 2.1]). *Let E be the total space of the rational fibration $F \rightarrow E \rightarrow S^{2n+1}$ with an elliptic space F satisfying $\dim H^*(F; \mathbb{Q}) \leq 2^{e_0(F)}$. Then $\dim H^*(E; \mathbb{Q}) \leq 2^{e_0(E)}$.*

Proof of Theorem. Since a formal elliptic space Y is hyperformal, there is an isomorphism as algebras

$$H^*(Y; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_p] \otimes \wedge(y_1, \dots, y_q) / (h_1, \dots, h_p) \quad (p, q \geq 0),$$

where the elements h_i ($i = 1, \dots, p$) are written as $h_i = f_i + g_i$ with a regular sequence $f_1, \dots, f_p$ in $\mathbb{Q}[x_1, \dots, x_p]$ and elements g_i in the ideal generated by $y_1, \dots, y_q$ [2, p.576–577]. We regard the algebra $H^*(Y; \mathbb{Q})$ as the exterior algebra $\wedge(y_1, \dots, y_q)$ if $p = 0$. Thus Y has the rational homotopy type of the total space of a fibration

$$X \rightarrow E_q \rightarrow S^{|y_1|} \times \dots \times S^{|y_q|},$$

where X is the F_0 -space with $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_p] / (f_1, \dots, f_p)$ and $S^{|y_i|}$ is the $|y_i|$ -dimensional sphere. If $q > 0$, by using Lemma 2.1 with [4, p.388, Example 4] inductively for fibrations

$$\begin{aligned} X &\rightarrow E_1 \rightarrow S^{|y_1|} \\ &\vdots \\ E_i &\rightarrow E_{i+1} \rightarrow S^{|y_{i+1}|} \\ &\vdots \\ E_{q-1} &\rightarrow E_q \rightarrow S^{|y_q|}, \end{aligned}$$

we have $\dim H^*(E_i; \mathbb{Q}) \leq 2^{\text{cup}_0(E_i)}$ for $i = 1, \dots, q$. Here each E_i is an elliptic space with

$$H^*(E_i; \mathbb{Q}) \cong \mathbb{Q}[x_1, \dots, x_p] \otimes \wedge(y_1, \dots, y_i) / (\bar{h}_1, \dots, \bar{h}_p),$$

where $\bar{h}_j$ is the obvious projection of h_j . □

3. TORAL RANK VS LS CATEGORY

Let X be a simply connected finite cell complex. Recall that the toral rank of a space X , $\text{rk}(X)$, is the largest integer n such that an n -torus can act continuously on X with all its isotropy subgroups finite. In [7], S. Halperin conjectured that $2^{\text{rk}(X)} \leq \dim H^*(X; \mathbb{Q})$, which gives an upper bound for toral rank. We compare the two bounds around formal elliptic spaces in the following table. Refer [4, Part II and Section 32] for the Sullivan minimal model theory.

X : elliptic		$2^{\mathrm{rk}(X)} \leq \dim H^*(X; \mathbb{Q})$	$\dim H^*(X; \mathbb{Q}) \leq 2^{\mathrm{cat} X}$
$\pi_{\mathrm{even}}(X) \otimes \mathbb{Q} = 0$		? (a)	yes ([9])
X : formal	$\chi_{\pi}(X) = 0$	yes (since $\mathrm{rk}(X) = 0$)	yes (Lemma)
	$\chi_{\pi}(X) < 0$	yes (b)	yes (Corollary)
X : pure		yes ([7, Proposition 1.5])	? (c)

Here X is called pure [4, p.435] if the differential d satisfies $dV^{\mathrm{even}} = 0$ and $dV^{\mathrm{odd}} \subset \wedge V^{\mathrm{even}}$ for the Sullivan minimal model $\mathcal{M}(X) = (\wedge V, d)$ of X , where $V = \bigoplus_{i \geq 1} V^i$ with $\dim V^i = \dim \pi_i(X) \otimes \mathbb{Q}$. For example, a homogeneous space is pure. Note a pure space with $\dim V^{\mathrm{even}} = \dim V^{\mathrm{odd}}$ is an F_0 -space.

(a) is “yes” if X is a space of two-stage Sullivan minimal model and coformal, i.e., V decomposes as $V \cong U \oplus W$ with $dU = 0$, $dW \subset \wedge U$ and d is quadratic, respectively [1, Proposition 3.1].

(b) is obtained from $\mathrm{rk}(X) \leq -\chi_{\pi}(X) = -(\dim V^{\mathrm{even}} - \dim V^{\mathrm{odd}}) = q$ [7, Theorem 1.1] and $\dim H^*(X; \mathbb{Q}) \geq 2^q$ when $H^*(X; \mathbb{Q})$ is given by $\mathbb{Q}[x_1, \dots, x_p] \otimes \wedge(y_1, \dots, y_q)/(h_1, \dots, h_p)$, where $\mathcal{M}(X) = (\wedge(x_1, \dots, x_p, y_1, \dots, y_q, v_1, \dots, v_p), d)$ with $dx_i = dy_i = 0$, $dv_i = h_i$.

(c) is “yes” if $\dim V^{\mathrm{even}} = 1$, because then there is an isomorphism $H^*(X; \mathbb{Q}) \cong \mathbb{Q}[x]/(x^n) \otimes \wedge(y_1, \dots, y_q)$.

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