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JORDAN LEFT DERIVATIONS ON SEMIPRIME RINGS

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Throughout, R will represent an associative ring. As usual we write $[x, y] = xy - yx$ for all $x, y \in R$. Let $D: R \rightarrow R$ be an additive mapping. D is called a derivation (resp. a left derivation) if $D(xy) = D(x)y + xD(y)$ (resp. $D(xy) = xD(y) + yD(x)$) holds for all $x, y \in R$. And D is called a Jordan derivation (resp. a left Jordan derivation) in case $D(x^2) = D(x)x + xD(x)$ (resp. $D(x^2) = 2xD(x)$) for all $x \in R$. Obviously, every derivation is a Jordan derivation. But in general, the converse is not true. A well known result of Herstein [6] states that every Jordan derivation on a prime ring R of $\text{char} \neq 2$ is a derivation. A brief proof of this result can be found in [1]. Cusack has generalized this result to 2-torsion free semiprime rings [4] (see also [2]). One can easily prove that in a noncommutative prime ring any left derivation is zero. Moreover, Brešar and the author have proved that the existence of a nonzero Jordan left derivation on a prime ring R of $\text{char} \neq 2, 3$ forces R to be commutative. This result can be considered as an extension of the well known result of Posner [9]. It should be mentioned that the result of [3] concerning Jordan left derivations has been improved by Deng [5]. In [3] one can find an example which shows that in a noncommutative semiprime ring there exists a nonzero Jordan left derivation.

In [8], Giambruno and Herstein have proved that if D is a derivation on a semiprime ring R such that for some positive integer n the relation $D(x)^n = 0$ holds for all $x \in R$, then $D = 0$. It is well known that if D and G are derivations on a 2-torsion free semiprime ring R , such that $D^2(x) = G(x)$ holds for all $x \in R$, then $D = 0$.

Our present objective is to prove the following theorems which modify the results noting about to Jordan left derivations on semiprime rings.

Theorem 1. *Let R be a 2-torsion free semiprime ring and $D: R \rightarrow R$ a Jordan left derivation. If there exists a positive integer n such that $D(x)^n = 0$ for all $x \in R$, then $D = 0$.*

Theorem 2. *Let R be a 2-torsion free and 3-torsion free semiprime ring. If R admits Jordan derivations D and $G: R \rightarrow R$ such that $D^2(x) =$*

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$G(x)$ for all $x \in R$, then $D = 0$.

In preparation for proving our theorems, we state the following three lemmas.

Lemma 3. *Let R be a 2-torsion free ring. If $D: R \rightarrow R$ is a Jordan left derivation then for all $x, y \in R$:*

- (a) $D(xy + yx) = 2xD(y) + 2yD(x)$,
- (b) $D(xy) = x^2D(y) + 3xyD(x) - yxD(x)$.

Proof. A special case of Proposition 1.1 in [3].

Lemma 4. *Let R be a 2-torsion free and 3-torsion free ring, and $D: R \rightarrow R$ a Jordan left derivation. If $D([D(x), x], x) = 0$ holds for all $x \in R$, then $[D(x), x]D(x) = 0$ is fulfilled for all $x \in R$.*

Proof. Using Lemma 3 we obtain

$$\begin{aligned} 0 &= D([D(x), x], x) \\ &= D(D(x)x^2 + x^2D(x)) - 2D(xD(x)x) \\ &= 4D(x)xD(x) + 2x^2D^2(x) - 2x^2D^2(x) - 6xD(x)^2 + 2xD(x)^2 \\ &= 6[D(x), x]D(x). \end{aligned}$$

Thus we have $6[D(x), x]D(x) = 0$, which completes the proof, since R is 2-torsion free and 3-torsion free.

Lemma 5. *Let R a noncommutative prime ring of characteristic different from two, and let $D: R \rightarrow R$ be a left Jordan derivation. In this case $D = 0$.*

Proof. A special case of Theorem 1 in [5].

Proof of Theorem 1. Since R is semiprime, $\bigcap P = (0)$ where the intersection runs over all prime ideals of R . We intend to prove that $D(P) \subset P$ for every prime ideal P of R . Let $a \in P$, $x \in R$; then

$$0 = D(ax + xa)^n = 2^n(aD(x) + xD(a))^n,$$

which gives

$$(aD(x) + xD(a))^n = 0,$$

since R is 2-torsion free. In other words

$$(xD(a))^n \equiv 0 \pmod{P}.$$

Thus, in the prime ring $R' = R/P$, we have

$$(x'D(a'))^n = 0 \quad \text{for all } x' \in R'.$$

By a well known result of Levitzki (see Lemma 1.1 in [7]) we have $x'D(a)' = 0$ for all $x' \in R'$, whence it follows $D(a)' = 0$ since R' is prime. In other words $D(a) \in P$, and so $D(P) \subset P$. Therefore $D(P) \subset P$ for all prime ideals P of R , and so D induces a left Jordan derivation D' on the prime ring $R' = R/P$. Let us first assume that R' is commutative. In this case D' is derivation and we have also $D'(x')^n = 0$, whence it follows $D' = 0$. In case R' is noncommutative it follows by Lemma 5 that $D' = 0$. Thus, in any case $D'(R') = (0)$, that is, $D(R) \subset P$ for all prime ideals P of R . Since $\bigcap P = (0)$, we obtain $D(R) = (0)$, hence $D = 0$. The proof of the theorem is complete.

Proof of Theorem 2. Putting x^2 for x in $D^2(x) = G(x)$ we obtain

$$(1) \quad D(xD(x)) = xG(x), \quad x \in R.$$

Let us prove that for all $x \in R$,

$$(2) \quad D(D(x)x) = 2D(x)^2 + xG(x).$$

Using Lemma 3 and (1) we obtain

$$\begin{aligned} D(D(x)x) &= D(D(x)x + xD(x)) - D(xD(x)) \\ &= 2D(x)^2 + 2xD^2(x) - xG(x) \\ &= 2D(x)^2 + xG(x). \end{aligned}$$

From (1) and (2) we obtain

$$(3) \quad D([D(x), x]) = 2D(x)^2, \quad x \in R.$$

The linearization of (3) gives

$$D([D(x), y] + [D(y), x]) = 2D(x)D(y) + 2D(y)D(x), \quad x, y \in R.$$

Putting in the above relation $y = x^2$, using Lemma 3 and (3), we obtain

$$\begin{aligned} 0 &= D([D(x), x^2] + [D(x^2), x]) - 2D(x)D(x^2) - 2D(x^2)D(x) \\ &= D([D(x), x]x + x[D(x), x]) \\ &\quad + 2D(x[D(x), x]) - 4D(x)xD(x) - 4xD(x)^2 \\ &= 2[D(x), x]D(x) + 2xD([D(x), x]) \\ &\quad + 2D(x[D(x), x]) - 4D(x)xD(x) - 4xD(x)^2 \\ &= 2D(x)xD(x) - 2xD(x)^2 + 4xD(x)^2 \\ &\quad + 2D(x[D(x), x]) - 4D(x)xD(x) - 4xD(x)^2 \\ &= -2D(x)xD(x) - 2xD(x)^2 + 2D(x[D(x), x]). \end{aligned}$$

Thus we have

$$(4) \quad D(x[D(x), x]) = D(x)xD(x) + xD(x)^2, \quad x \in R.$$

Let us Prove the identity

$$(5) \quad D([D(x), x]x) = D(x)xD(x) + xD(x)^2, \quad x \in R.$$

Using Lemma 3 and (3) we have

$$\begin{aligned} D([D(x), x]x + x[D(x), x]) &= 2[D(x), x]D(x) + 2xD([D(x), x]) \\ &= 2[D(x), x]D(x) + 4xD(x)^2. \end{aligned}$$

Now applying (4) we obtain

$$\begin{aligned} D([D(x), x]x) &= 2[D(x), x]D(x) + 4xD(x)^2 - D(x[D(x), x]) \\ &= 2[D(x), x]D(x) + 4xD(x)^2 - D(x)xD(x) - xD(x)^2 \\ &= D(x)xD(x) + xD(x)^2, \end{aligned}$$

which completes the proof of (5).

From (4) and (5) we obtain

$$(6) \quad D([D(x), x], x) = 0, \quad x \in R.$$

From (6) and Lemma 4 it follows

$$(7) \quad [D(x), x]D(x) = 0, \quad x \in R,$$

Using (7) and (3) we obtain

$$\begin{aligned} D(D(x)[D(x), x]) &= D(D(x)[D(x), x] + [D(x), x]D(x)) \\ &= 2D(x)D([D(x), x]) + 2[D(x), x]G(x). \end{aligned}$$

Thus we have

$$(8) \quad D(D(x)[D(x), x]) = 4D(x)^3 + 2[D(x), x]G(x), \quad x \in R.$$

Let us prove the relation

$$(9) \quad D(D(x)[D(x), x]) = -6[D(x), x]G(x), \quad x \in R.$$

Using (7) and Lemma 3 we obtain

$$\begin{aligned} 0 &= D([D(x), x]D(x)) \\ &= D(D(x)x D(x)) - D(x D(x)^2) \\ &= D(x)^3 + 3D(x)x G(x) - x D(x)G(x) - D(x D(x)^2). \end{aligned}$$

Thus we have

$$(10) \quad D(x D(x)^2) = D(x)^3 + 3D(x)x G(x) - x D(x)G(x), \quad x \in R.$$

Now we have

$$D(D(x)^2 x + x D(x)^2) = 2D(x)^3 + 4x D(x)G(x), \quad x \in R.$$

From the above relation and (10) it follows

$$(11) \quad D(D(x)^2 x) = D(x)^3 + 5x D(x)G(x) - 3D(x)x G(x), \quad x \in R.$$

From (10) and (11) we obtain

$$D([D(x)^2, x]) = 6[x, D(x)]G(x).$$

Thus we have according to (7),

$$\begin{aligned} 6[x, D(x)]G(x) &= D([D(x)^2, x]) \\ &= D([D(x), x]D(x) + D(x)[D(x), x]) \\ &= D(D(x)[D(x), x]), \end{aligned}$$

which completes the proof of (9). Combining (8) with (9) we arrive at

$$(12) \quad D(x)^3 + 2[D(x), x]G(x) = 0, \quad x \in R$$

Now starting from (7) and using Lemma 3 we obtain

$$\begin{aligned} 0 &= D(D(x)[D(x), x]D(x)) \\ &= D(x)^2 D([D(x), x]) + 3D(x)[D(x), x]D^2(x) - [D(x), x]D(x)D^2(x) \\ &= 2D(x)^4 + 3D(x)[D(x), x]G(x). \end{aligned}$$

Thus we have

$$(13) \quad 2D(x)^4 + 3D(x)[D(x), x]G(x) = 0, \quad x \in R$$

From (12) and (13) it follows $D(x)^4 = 0$, $x \in R$, which completes the proof of the theorem, since all the assumptions of Theorem 1 are fulfilled.

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