

Mathematical Journal of Okayama University

Volume 39, Issue 1

1997

Article 14

JANUARY 1997

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Math. J. Okayama Univ. **39** (1997), 159–163

THE TWISTOR SPACE OF DISTRIBUTIONS

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1. Introduction. In this paper a naturally defined kind of distributions on the twistor space of distributions over some manifold N is established. It is defined by an ordered pair (D, ω) of a distribution D on the base manifold N and a linear connection ω on it. Concerning integrability of this distribution (denoted by D^ω) it is shown that cases $\dim D^\omega = 2$ and $\dim D^\omega = \dim N$ are distinguished. Nevertheless D^ω is integrable very rare in any case (see theorems 2.2 and 2.4). Moreover consequences of induced action of a gauge transformation on this distribution are also considered. One works in the C^∞ category.

2. Constructions. Let $\pi_L: L(N) \rightarrow N$ be the principal $GL(n; \mathbb{R})$ bundle of linear frames over an $n = p + q$ -dimensional manifold N . The twistor space of p -dimensional distributions over N is constructed as follows: for the subgroup

$$(2.1) \quad G = \left\{ \begin{pmatrix} A & B \\ O & C \end{pmatrix} \in GL(n; \mathbb{R}) \mid A \in GL(p; \mathbb{R}), C \in GL(q; \mathbb{R}) \right\}$$

any reduced G -subbundle $Q \subset L(N)$ corresponds to some p -dimensional distribution $D \subset TN$ on the manifold N , [3]. There exists a 1:1 correspondence between such subbundles and the set of sections of a fiber bundle associated with $L(N)$ whose standard fiber is the space $GL(n; \mathbb{R})/G$. This fiber bundle can be naturally identified with the quotient $L(N)/G$; and this quotient represents the twistor space Z of p -dimensional distributions over manifold N . This generalized framework is considered in [1] and [4] in another setting.

Fibrations are exposed by

$$(2.2) \quad L(N) \xrightarrow{\pi_L} Z = L(N)/G \xrightarrow{\pi} N.$$

Thus any p -dimensional distribution $D \subset TN$ on N uniquely determines a global section (also denoted by) $D: N \rightarrow Z$ which further defines a reduced G -subbundle $Q := \pi_Z^{-1}(D(N))$. By definition some linear connection ω

on $L(N)$ is reducible to Q iff its horizontal distribution $H^\omega \subset TL(N)$ is tangent to Q . Equivalently, ω is reducible to Q iff

$$(2.3) \quad (\pi_Z)_*(H^\omega|_Q) = D_*(TN)$$

holds. In the same manner a horizontal distribution on the space Z is defined by

$$(2.4) \quad (\pi_Z)_*(H^\omega) \subset TZ.$$

Here ω need not be reducible to Q . Consequently if $D^H \subset H^\omega$ is the horizontal lift of a distribution $D \subset TN$ defined by the connection ω on $L(N)$, then a distribution $D^\omega \subset TZ$ is defined on the space Z as

$$(2.5) \quad D^\omega := (\pi_Z)_*(D^H).$$

Before considering the question of integrability of this distribution first will be proved the following very useful

Lemma 2.1. *Let B_i be the standard horizontal vector fields on bundle $L(N)$ defined by a connection ω and basis vectors $u_i \in \mathbb{R}^n$ ($i = 1, \dots, n$), [2]. Denote by Ω and T the curvature and the torsion of ω respectively (viewed on $L(N)$). Then the following commutator formula holds.*

$$(2.6) \quad [B_i, B_j] = -T^k(B_i, B_j)B_k - \Omega_i^k(B_i, B_j)E_k^l.$$

Here (E_k^l) is the standard basis in $\mathfrak{gl}(n; \mathbb{R})$, and one identifies (i.e., uses the same letter) elements from $\mathfrak{gl}(n; \mathbb{R})$ and corresponding fundamental fields on $L(N)$ (a convention is $v_1 \wedge v_2 := v_1 \otimes v_2 - v_2 \otimes v_1$).

Proof. Let M be a manifold and (e_α) and (ε^α) some frame with corresponding coframe on M respectively. Then holds the following relations ([5, p. 26])

$$(2.7) \quad \begin{aligned} [e_\alpha, e_\beta] &= C_{\alpha\beta}^\gamma e_\gamma, \\ d\varepsilon^\gamma &= -\frac{1}{2}C_{\alpha\beta}^\gamma \varepsilon^\alpha \wedge \varepsilon^\beta. \end{aligned}$$

Here C -symbols are defined by the choice of frame (e_α) . Let

$$(2.8) \quad \begin{aligned} d\theta^i &= -\omega_j^i \wedge \theta^j + \frac{1}{2}T_{jk}^i \theta^j \wedge \theta^k \\ d\omega_j^i &= -\omega_k^i \wedge \omega_j^k + \frac{1}{2}\Omega_{jkl}^i \theta^k \wedge \theta^l \end{aligned}$$

be the structure equations of the connection ω . Here $\Omega_{jkl}^i := \Omega_j^i(B_k, B_l)$, $T_{jk}^i := T^i(B_j, B_k)$, and (θ^i) are components of the canonical 1-form on $L(N)$. Because $(\theta^i; \omega_j^i)$ is a (global) coframe on $L(N)$ which corresponds to the frame $(B_i; E_j^i)$, by comparing (2.7) and (2.8) one obtains (2.6).

Let X^ω, Y^ω and $[X, Y]^\omega$ be horizontal lifts to $L(N)$ of vector fields X, Y and $[X, Y]$ on N respectively which are defined by the connection ω on $L(N)$. Then by using (2.6) one finds the following relation

$$(2.9) \quad [X^\omega, Y^\omega] = [X, Y]^\omega - T^k(X^\omega, Y^\omega)B_k - \Omega(X^\omega, Y^\omega).$$

By this formula one is able to prove the following

Theorem 2.2. *Let $D \subset TN$ be a p -dimensional foliation on a manifold N endowed with a torsionfree connection ω by which the distribution $D^\omega \subset TZ$ introduced in (2.5) is defined. Suppose that vector fields Y_1 and Y_2 take value in D while Y_3 is any vector field on N . Then the distribution D^ω will be integrable if*

$$(2.10) \quad R(Y_1, Y_2)Y_3 = fY_3$$

holds. Here R is the curvature tensor of ω while f is a function on N which does not depend on Y_3 .

Proof. From (2.9) it follows that D^ω will be integrable iff $\Omega(X^\omega, Y^\omega)$ takes value in $\mathfrak{S}$, the Lie algebra of G , when X and Y take value in D . But fields X^ω and Y^ω are right $\text{GL}(n; \mathbb{R})$ -invariant, consequently

$$(2.11) \quad \Omega_{pg}(X^\omega, Y^\omega) = \text{Ad}(g^{-1})\Omega_p(X^\omega, Y^\omega)$$

holds for all $p \in L(N)$, $g \in \text{GL}(n; \mathbb{R})$. Thus in order that D^ω is integrable $\Omega(X^\omega, Y^\omega)$ must take value in an ideal of $\mathfrak{gl}(n; \mathbb{R})$ which is contained in $\mathfrak{S}$, e.g.,

$$(2.12) \quad \Omega(X^\omega, Y^\omega) = \hat{f}I$$

can hold. where $\hat{f} \in C^\infty(L(N))$. This condition is expressed on the level of N in a form given in (2.10).

Because it is supposed that ω is a torsionfree connection, the first Bianchi identity

$$(2.13) \quad R(Y_1, Y_2)Y_3 + R(Y_2, Y_3)Y_1 + R(Y_3, Y_1)Y_2 = 0$$

holds. Hence if in (2.10) also $Y_3 \in D$ and moreover Y_1, Y_2, Y_3 are linearly independent, then from (2.13) it follows that $f \equiv 0$. Thus in the setting of Theorem 2.2 we have

Corollary 2.3. *If $\dim D > 2$, then D^ω is integrable if ω is flat.*

On the other hand, from (2.13) and (2.10) it does not follow that there are no nontrivial examples (i.e., ω is not flat) of integrable D^ω in case of $\dim D = 2$. Important special case is $\dim D = \dim N$, i.e., $D^\omega = (\pi_Z)_* H^\omega$ from (2.4): here there is no need to impose condition that a connection ω is torsionfree because the term $T^k(X^\omega, Y^\omega)B_k$ in (2.9) is not troublesome at all now. Consequently the first Bianchi identity does not make restrictions so that the following holds.

Theorem 2.4. *The horizontal distribution D^ω on the twistor space Z which is introduced in (2.4) is integrable if*

$$(2.14) \quad R(Y_1, Y_2)Y_3 = fY_3$$

holds for any vector fields Y_i on N . Here the function f on N does not depend on Y_3 .

One can notice that if ω is a torsionfree connection on $L(N)$ which is reducible to a G -subbundle $Q \subset L(N)$ defined by a foliation $D \subset TN$, then the distribution $D^\omega|_{D(N)}$ on the submanifold $D(N)$ is integrable.

3. On gauge-action upon distributions. Let $f: L(N) \rightarrow L(N)$ be any gauge-transformation in the bundle $L(N)$. Because its fiber-preserving action commutes with the right action of the group $GL(n; \mathbb{R})$ on $L(N)$, it follows that f induces in a natural manner a transformation (also denoted by f) on the twistor space Z . Consequently f maps any distribution $D: N \rightarrow Z$ into another fD . Then $fD = D$ holds iff f preserves the G -structure $Q \subset L(N)$ which corresponds to D . On the other hand f acts naturally upon $TL(N)$ and further upon TZ . In this setting we have

Theorem 3.1. *If the subbundle $Q \subset L(N)$ which corresponds to a distribution $D \subset TN$ is left invariant by a gauge-transformation f , and some connection ω on $L(N)$ is reducible to Q , then the distribution $D^\omega|_{D(N)} \subset TD(N)$ is preserved under the action of f upon TZ .*

Proof. A sufficient condition that f preserves the distribution $D^H \subset TL(N)$ is that it preserves the connection ω , i.e., its horizontal distribution (if $D^H \equiv H^\omega$, i.e., $D \equiv TN$, this condition is also necessary). But if $fD = D$ and ω is reducible to Q then $f^*\omega$ is also reducible to Q . Consequently by (2.3)

$$(\pi_Z)_*(H^\omega|_Q) = (\pi_Z)_*(H^{f^*\omega}|_Q)$$

where from follows the statement of the theorem.

From this Theorem it follows that if $fQ = Q$ and ω is reducible to Q , then

$$D^{f^*\omega}|_{D(N)} = D^\omega|_{D(N)}.$$

From this does not follows that $D^{f^*\omega} = D^\omega$ on the whole space Z . A sufficient condition for it is, say, that $f^*\omega - \omega$ take value in the center $\{cI \mid c \in \mathbb{R}\}$ of $\mathfrak{gl}(n; \mathbb{R})$.

REFERENCES

- [1] L. B. BERGERY and T. OCHIAI: On some generalization of the construction of twistor spaces, L.M.S. Symposium on Global Riemannian Geometry, Durham, 1982.
- [2] S. KOBAYASHI and K. NOMIZU: Foundations of Differential Geometry, vol. I and II, Interscience Publ., 1963–1969.
- [3] S. KOBAYASHI: Transformation Groups in Differential Geometry, Springer-Verlag, 1972.
- [4] N. O'BRIAN and J. RAWNSLEY: Twistor spaces, Annals of Global Analysis and Geometry **3** (1985), 29–58.
- [5] K. YANO and M. KON: Structures on Manifolds, World Scientific, 1984.

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(Received March 17, 1997)