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On connected locally connected spaces with cut points

Iwao Yoshioka*

*Okayama University

ON CONNECTED LOCALLY CONNECTED SPACES WITH CUT POINTS

Dedicated to Professor TAKESHI INAGAKI on his 60th birthday

IWAO YOSHIOKA

If M is a connected space and p is a point of M such that $M-p$ is not connected, then p will be called a *cut point* of M [1]. If p is a cut point, $M-p$ is the sum of two mutually separated sets $M_1(p)$ and $M_2(p)$, then $M_1(p)$ and $M_2(p)$ will be called *sects* of M from p [2]. A point q is said to be separated from a point r by p if there exists a separation

$$M-p=M_1(p)\cup M_2(p),\ \text{where } M_1(p)\ni q,\ M_2(p)\ni r.$$

If r is a point of M, then $S(r, M)$ denotes the set of all points which are separated from r by at least one cut point of M, and $T(r, M)$ the set of all cut points of M except the point r. The boundary of a point set A will be denoted by $Bd\ A$. In this note, all spaces are connected locally connected Hausdorff spaces with at least one cut point. The purpose of this note is to study the conditions under which the set $S(r, M)$ is connected.

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Lemma 1. *If p is a point of a space M, then $Bd\ S(p, M)\subset \overline{T(p, M)}$.*

Proof. Let us assume the contrary and let $x\in Bd\ S(p, M)-\overline{T(p, M)}$. Then, since M is locally connected, there exist neighbourhoods $U(x)$, $V(x)$ of a point x such that

$$x\in V(x)\subset C\subset U(x) \text{ and } U(x)\cap T(p, M)=\emptyset,$$

where C is a connected set.

On the other hand, there is a point $y\in V(x)\cap S(p, M)$. Therefore, we can take a point $q\in T(p, M)$ and have a separation

$$M-q=M_1(q)\cup M_2(q),\ \text{where } M_1(q)\ni p,\ M_2(q)\ni y.$$

Since $C\ni y$ and $C\not\ni q$, $x\in M_2(q)\subset S(p, M)$. This contradiction completes the proof.

Lemma 2. *Let p be a point of a space M. If $S(p, M)$ is connected and $Bd\ S(p, M) \cap T(p, M) \neq \emptyset$, then $Bd\ S(p, M)$ is a single point.*

Proof. Let q be a point of the set $Bd\ S(p, M) \cap T(p, M)$.
Then we have a separation

$$M-q=M_1(q)\cup M_2(q), \text{ where } p\in M_1(q),\ M_2(q)\subset S(p, M).$$

Since $q\in Bd\ S(p, M)$, $S(p, M)\subset M-q$. Therefore, we have $M_2(q)=S(p, M)$, because $M_2(q)$ is open and closed in $M-q$. This means that $Bd\ S(p, M)$ is a single point.

Theorem 3. *For a point p of a space M, the following four conditions are equivalent* :

(a) *If $S(p, M)$ is non-vacuous, it is a connected set.*

(b) *If $M_2(q)$, $M_2(q')$ are two sects of M from q, q' not containing p, where q, $q'\in T(p, M)$, then there exists a connected sect of M, not containing p, which contains $M_2(q)\cup M_2(q')$.*

(c) *If $M_2(q)$, $M_2(q')$ are two connected sects of M from q, q' not containing p, where q, $q'\in T(p, M)$ then there exists a connected sect of M, not containing p, which contains $M_2(q)\cup M_2(q')$.*

(d) *$Bd\ S(p, M)$ is a single point such that if $Bd\ S(p, M)\neq p$, M is uniquely separated by $Bd\ S(p, M)$, and if $Bd\ S(p, M)=p$, there exists a component K of $S(p, M)$ with $\overline{T(p, M)-K} \not\ni Bd\ S(p, M)$.*

Proof. (a) implies (b): Let $M_2(q)$, $M_2(q')$ be sects satisfying the assumption of (b) and put

$$\mathfrak{A}=\left\{N_2(t)\mid M_2(q)\subset N_2(t)\right\},\quad \mathfrak{A}^*=\bigcup_{N_2(t)\in\mathfrak{A}} N_2(t)$$

where $N_2(t)$ is any sect of M from t not containing p for each point $t\in T(p, M)$. Let us assume that $x\in S(p, M)-\mathfrak{A}^*$. Then, there exists a cut point u with a separation

$$M-u=H_1(u)\cup H_2(u), \text{ where } H_1(u)\ni p,\ H_2(u)\ni x.$$

Then $H_2(u)\cap\mathfrak{A}^*=\emptyset$. For, if $H_2(u)\cap\mathfrak{A}^*\neq\emptyset$, there exists $K_2(t)\in\mathfrak{A}$ and $K_2(t)\cap H_2(u)\neq\emptyset$. If $u=t$, then $H_2(u)\cup K_2(t)\in\mathfrak{A}$. This implies $u\neq t$. Since $H_1(u)\cap K_1(t)\ni p$,

$$H_2(u)\subset K_2(t) \text{ or } H_2(u)\supset K_2(t).$$

Hence $x \in \mathfrak{A}^*$. This implies $H_2(u) \cap \mathfrak{A}^* = \emptyset$. By the connectedness of $S(p, M)$, $S(p, M) = \mathfrak{A}^*$, because $\mathfrak{A}^*$ is open and closed in $S(p, M)$. Therefore, there exists a sect $N_2(t) \in \mathfrak{A}$ such that $N_2(t) \cap M_2(q') \neq \emptyset$. Then

$$(1) \qquad N_2(t) \cup M_2(q') \supset M_2(q) \cup M_2(q') \text{ if } t = q',$$

$$(2) \qquad M_2(q') \supset M_2(q) \cup M_2(q') \text{ or } N_2(t) \supset M_2(q) \cup M_2(q') \text{ if } t \neq q'.$$

(1) and (2) imply that there exists a sect $L_2(s) \in \mathfrak{A}$ with $L_2(s) \supset M_2(q) \cup M_2(q')$. If there exists a sect $H_2(r) \in \mathfrak{A}$ with $H_2(r) \ni s$, we consider a component C of $H_2(r)$ containing s. Then

$$C \supset L_2(s) \supset M_2(q) \cup M_2(q').$$

If there exists no such sect, then $s \notin S(p, M)$. Therefore, by the connectedness of $S(p, M)$,

$$L_2(s) = S(p, M) \supset M_2(q) \cup M_2(q').$$

It is clear that (b) implies (c) and (c) implies (a).

(a) implies (d): Suppose that $Bd\ S(p, M)$ contains distinct points t_1 and t_2. By Lemma 2,

$$(3) \qquad Bd\ S(p, M) \cap T(p, M) = \emptyset.$$

As M is a Hausdorff space, there exist neighbourhoods $U_i \ni t_i$ $(i=1, 2)$ with $U_1 \cap U_2 = \emptyset$. By the locally connectedness of M, there exist neighbourhoods V_i and connected sets C_i satisfying $U_i \supset C_i \supset V_i \ni t_i$ $(i=1, 2)$. By Lemma 1, there exist $q_i \in V_i \cap T(p, M)$ $(i=1, 2)$. Then $q_i \in S(p, M)$ $(i=1, 2)$ by $T(p, M) \subset \overline{S(p, M)}$ and (3). Let $H_2(u_i)$ be connected sects of M from u_i not containing p with $u_i \in T(p, M)$ and $q_i \in H_2(u_i)$ $(i=1, 2)$. Since (a) and (c) are equivalent, there exists a connected sect $H_2(u_3)$ of M from $u_3 \in T(p, M)$ not containing p such that $H_2(u_1) \cup H_2(u_2) \subset H_2(u_3)$. Hence

$$H_2(u_3) \cap C_1 \neq \emptyset \neq H_2(u_3) \cap C_2.$$

Then $C_1 \cap C_2 \ni u_3$, because C_1 and C_2 are connected and $H_2(u_3)$ is open and closed in $M - u_3$. This implies that $Bd\ S(p, M)$ is a single point. Now, let $t = Bd\ S(p, M)$. If $t \neq p$, there exists a separation such that

$$M - t = \{M - \overline{S(p, M)}\} \cup S(p, M), \text{ where } M - \overline{S(p, M)} \ni p.$$

This separation is uniquely determined. On the other hand, if $t = p$, then $\overline{T(p, M) - S(p, M)} = \emptyset$ from $T(p, M) \subset S(p, M)$.

(d) implies (a): Let $t = Bd\ S(p, M)$. Then $t \in \overline{T(p, M)}$ by Lemma 1.

If $t \neq p$, by assertion of (d), there exists a unique separation

$$M-t=\{M-\overline{S(p, M)}\}\cup S(p, M), \text{ where } M-\overline{S(p, M)}\ni p.$$

Hence, $S(p, M)$ is a connected set. Therefore, we can assume $t=p$. Then, there exists a component K of $S(p, M)$ such that

$$\text{(4)} \qquad \overline{T(p, M)-K}\not\ni t.$$

Since $T(p, M)\subset\overline{S(p, M)}$ and $t\notin T(p, M)$,

$$\text{(5)} \qquad T(p, M)-K=T(p, M)\cap\{S(p, M)-K\}.$$

Let us assume that $S(p, M)$ is not connected. Then, there exists a component L of $S(p, M)$ different from K. It follows from (5) that

$$\text{(6)} \qquad \overline{T(p, M)-K}\supset\overline{L\cap T(p, M)}.$$

By (4) and (6), there exist neighbourhoods U, V of t and a connected set C satisfying $U\supset C\supset V\ni t$ and

$$\text{(7)} \qquad U\cap L\cap T(p, M)=\emptyset.$$

There exists a point $z\in V\cap L$ by $Bd\ L=t$. Let $H_2(q)$ be a connected sect of M not containing p with $H_2(q)\ni z$, where $q\in T(p, M)$. To see $q\in L$, let $q\notin L$. Then $H_2(q)=L$, because L is a component of $S(p, M)$. Hence $t=BdL=Bd\ H_2(q)=q$. This contradiction implies $q\in L$. By (7) and $q\in L\cap T(p, M)$, $C\not\ni q$. Since $C\cap H_2(q)\ni z$, $H_2(q)\supset C\ni t$. This contradiction completes the proof.

Theorem 4. *Let M be a space. In order that all non-empty $S(p, M)$ $(p\in M)$ be connected, it is necessary and sufficient that M have one and only one cut point by which M is uniquely separated.*

Proof. Sufficiency: Let t be the unique cut point of the space M, and $M-t=M_1(t)\cup M_2(t)$ the separation by t. We have $p\neq t$ for non-empty $S(p, M)$ $(p\in M)$. Hence, if p is such a point belonging to $M_1(t)$, then $S(p, M)=M_2(t)$ is a connected set.

Necessity: Suppose that M contains two distinct cut points q_1, q_2. Let M_1 be a component of $M-q_1$ containing q_2, $M_2=M-(M_1\cup q_1)$, N_1 a component of $M-q_2$ containing q_1, and $N_2=M-(N_1\cup q_2)$. Then there exist two separations such that

$$\text{(8)} \qquad M-q_1=M_1\cup M_2, \quad M-q_2=N_1\cup N_2,$$

where $q_2\in M_1$, $q_1\in N_1$. Then

$$M_1 \cap N_1 \neq \emptyset \text{ and } M_2 \cap N_2 = \emptyset.$$

To see that $M_1 \cap N_1$ has a cut point of M, let us assume that $M_1 \cap N_1$ has no cut point of M. Let y be a point of $M_1 \cap N_1$ and let x be a point of $S(p, M)-(M_2 \cup N_2)$. Then there exist a cut point t of M and a separation

$$M-t=H_1(t) \cup H_2(t), \text{ where } y \in H_1(t), \ x \in H_2(t).$$

If $q_1=t$, then

$$H_1(t) \cup H_2(t)=M_1 \cup M_2.$$

Hence $x \in H_2(t) \subset M_2$ by the definition of M_2. This implies $q_1 \neq t$. Then, we have $t \in M_1$ by $q_1 \in H_1(t) \cup H_2(t)$. Analogously, $t \in N_1$. Therefore, we obtain that $S(y, M)=M_2 \cup N_2$ because $M_1 \cap N_1$ has no cut point. This contradicts the connectedness of $S(y, M)$.

Now, for an arbitrary separation

$$M-u=M_1(u) \cup M_2(u),$$

where u is any cut point of M in $M_1 \cap N_1$, we have

$$\text{(9)} \qquad \overline{M}_2 \subset M_1(u) \text{ or } \overline{M}_2 \subset M_2(u)$$

and

$$\text{(10)} \qquad \overline{N}_2 \subset M_1(u) \text{ or } \overline{N}_2 \subset M_2(u).$$

On the other hand, by the connectedness of $S(p, M)$ for each point $p \in M$, for a cut point u of M in $M_1 \cap N_1$ there exists no separation

$$M-u=M_1(u) \cup M_2(u)$$

which satisfies

$$\text{(11)} \qquad M_1(u) \supset M_2 \text{ and } M_2(u) \supset N_2$$

or

$$\text{(12)} \qquad M_1(u) \supset N_2 \text{ and } M_2(u) \supset M_2.$$

In fact, if for any cut point $u \in M_1 \cap N_1$ there exists a separation

$$M-u=M_1(u) \cup M_2(u)$$

satisfying (11) or (12), then $M_1(u) \supset M_2$ and $M_2(u) \supset N_2$. This is contrary to the connectedness of $S(u, M) \subset M-u$.

Now, let z be a cut point of M in $M_1 \cap N_1$ with the following separation

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$$M-z=M_1(z)\cup M_2(z).$$

Here, from (9)~(12) we may assume that $M_1(z)\supset \overline{M_3\cup N_2}$. Then

(13) $M_2(z)\subset S(q_1, M)$.

We shall see $q_2 \notin S(q_1, M)$. If not, there exist a cut point t and a separation

$$M-t=K_1(t)\cup K_2(t), \text{ where } K_1(t)\ni q_1,\ K_2(t)\ni q_2.$$

It follows that $K_1(t)\supset \overline{M}_2$ and $K_2(t)\supset \overline{N}_2$ by $M_1\cap N_1\ni t$. This contradicts (11) or (12). By (8) and the connectedness of $S(q_1, M)$, we have then

(14) $N_2=S(q_1, M)$.

However, since $N_2\subset M_1(z)$, (14) contradicts (13). This implies that M has a single cut point t of M. Hence, we have $Bd\ S(p, M)=t$ for each p in $M-t$ (Lemma 1) and M is uniquely separated by t (Theorem 3). This completes the proof.

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DEPERTMENT OF MATHEMATICS,
OKAYAMA UNIVERSITY

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