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## Group rings with solvable unit groups

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## GROUP RINGS WITH SOLVABLE UNIT GROUPS

Dedicated to Professor TAKESHI INAGAKI on the occasion  
of his sixtieth birthday

KAORU MOTOSE and HISAO TOMINAGA

Throughout,  $R$  will represent an artinian simple ring with 1,  $C$  the center of  $R$ , and  $G$  a finite group of order  $g$ .

Recently, in their paper [1], P. B. Bhattacharya and S. K. Jain proved the following : Let  $S$  be a right artinian ring with 1, and  $J(S)$  the (Jacobson) radical of  $S$ . Then, the unit group  $U(S)$  of  $S$  is solvable if and only if  $U(S/J(S))$  is solvable<sup>1)</sup>, namely,  $S/J(S)$  is a finite direct sum of fields,  $(\text{GF}(2))_2$  and  $(\text{GF}(3))_3$ . This result will be used freely in the sequel. The main theme of our discussion will concern the solvability of the unit group of the group ring  $RG$ .

For semi-simple group rings, we shall prove the following :

**Theorem 1.** *Assume that  $g$  is not divisible by the characteristic of  $C$ . Then,  $U(RG)$  is solvable if and only if there holds one of the following :*

- (i)  $R=C$  and  $G$  is abelian (or equivalently,  $U(RG)$  is nilpotent<sup>2)</sup>).
- (ii)  $R=(\text{GF}(2))_2$  and  $G=1$ .
- (iii)  $R=(\text{GF}(3))_2$  and  $G$  is an abelian group of exponent 2.
- (iv)  $R=\text{GF}(3)$  and  $G=G_1 \times G_2$ , where  $G_1$  is an abelian group of exponent 2 and  $G_2$  is a non-abelian indecomposable 2-group such that  $U(RG_2)$  is solvable.

*Proof.* Assume first  $U(RG)$  is solvable. Since  $U(R)$  is solvable,  $R=C$  or  $(\text{GF}(2))_2$  or  $(\text{GF}(3))_3$ . Now, we shall distinguish between three cases :

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1) More generally, there holds the following : Let  $S$  be a ring with 1, and  $N$  a nilpotent ideal of  $S$ . If  $U(S/N)$  is solvable then  $U(S)$  is solvable (and conversely). To see this, we consider the group homomorphism  $\alpha: U(S) \rightarrow U(S/N)$  induced by the natural ring homomorphism  $S \rightarrow S/N$ . Let  $(\text{Ker } \alpha)^{[n]}$  be the  $n$ -th commutator subgroup of  $\text{Ker } \alpha$ . Noting that  $a^{-1}b^{-1}ab-1 = a^{-1}b^{-1}((a-1)(b-1)-(b-1)(a-1))$  for  $a, b \in U(S)$ , we can easily see  $\{u-1 \mid u \in (\text{Ker } \alpha)^{[n]}\} \subset N^{2^n}$ , which implies the solvability of  $\text{Ker } \alpha$ .

2) Cf. [3].

Case 1.  $R = C \neq \text{GF}(3)$ : It suffices then to consider the case  $R = \text{GF}(2)$ :  $RG = F_1 \oplus \cdots \oplus F_k \oplus (\text{GF}(2))_2^{(g)}$ , where  $F_i$  are fields over  $\text{GF}(2)$  and  $(\text{GF}(2))_2^{(g)}$  means the direct sum of  $l$  copies of  $(\text{GF}(2))_2$ . Since  $\text{GL}(2, 2)$  is isomorphic to the symmetric group  $S_3$  and the alternative group  $A_3$  is abelian, we obtain a normal chain  $G = G_0 \triangleright G_1 \triangleright \cdots \triangleright G_{n-1} \triangleright G_n$  such that  $(G_{i-1} : G_i) \leq 2$  and  $G_n$  is abelian. Noting that  $g$  is odd, we readily see that  $G$  is equal to the abelian group  $G_n$ .

Case 2.  $R = (\text{GF}(2))_2$  or  $(\text{GF}(3))_2$ : Evidently,  $RG = F_1 \oplus \cdots \oplus F_k \oplus R^{(g)}$ . Since  $U(CG)$  is solvable and  $RG = (CG)_2$ , one will easily see that  $CG = C^{(g)}$ . If  $R = (\text{GF}(2))_2$  then  $U(CG) = 1$ , and hence  $G = 1$ . On the other hand, if  $R = (\text{GF}(3))_2$  then  $U(CG)$  is an abelian group of exponent 2, and hence so is  $G$ .

Case 3.  $R = \text{GF}(3)$ : Assume  $G$  is non-abelian. Then,  $G = G_1 \times G_2$ , where  $G_2$  is non-abelian and indecomposable. Since both  $U(RG_1)$  and  $U(RG_2)$  are solvable,  $RG_1 = F'_1 \oplus \cdots \oplus F'_{k'} \oplus (R)_2^{(l')}$  and  $RG_2 = F''_1 \oplus \cdots \oplus F''_{k''} \oplus (R)_2^{(l'')}$  where  $F'_i$  and  $F''_j$  are fields over  $R$  and  $l', l'' > 0$ . Accordingly,  $RG$  is the direct sum of  $\bigoplus_{i,j} (F'_i \otimes_R F''_j)$ ,  $\bigoplus_i (F'_i)^{(l')}$ ,  $\bigoplus_j (F''_j)^{(l'')}$  and  $(R)_2^{(l'l'')}$ . It follows then  $l' = 0$  and  $F'_i = R$ . Hence,  $G_1$  is an abelian group of exponent 2. Now, let  $G_2^*$  be the residue class group of  $G_2$  modulo its center. Since  $G_2^*$  can be regarded as a subgroup of the direct product of  $l''$  copies of  $\text{PGL}(2, 3)$  and  $\text{PGL}(2, 3) (\cong S_4)$  is of order  $2^3 \cdot 3$ ,  $G_2^*$  is a non-trivial 2-group, and hence  $G_2$  is nilpotent. Recalling here that  $G_2$  is indecomposable, we readily see that  $G_2$  is a 2-group.

Concerning the converse, it suffices to prove that if (iii) or (iv) holds then  $U(RG)$  is solvable. In fact, if there holds (iii) then  $CG = C^{(g)}$ , whence we readily see that  $RG = (C)_2^{(g)}$  and  $U(RG)$  is solvable. Next, if there holds (iv) then  $RG_i = R^{(g_i)}$  ( $g_i$  the order of  $G_i$ ) and  $U(RG) = U(RG_1 \otimes_R RG_2)$  is solvable.

**Remark.** Let  $R = \text{GF}(3)$ . If  $g = 8$  then  $U(RG)$  is solvable. In fact, if  $G$  is non-abelian then the commutator subgroup  $G'$  of  $G$  coincides with the center of  $G$  and  $G/G'$  is a 2-elementary abelian group of order 4. Accordingly,  $G$  possesses four 1 dimensional representations in  $R$ , and it is easy to see that  $RG = R^{(4)} \oplus (R)_2$ . However, in case  $g = 16$ , there exists an indecomposable  $G$  such that  $U(GR)$  is not solvable: Let  $G$  be the group  $\langle a, b \rangle$  with defining relations  $a^8 = 1$ ,  $b^2 = 1$ ,  $b^{-1}ab = a^7$ . To be easily seen,  $G$  is indecomposable and possesses the irreducible representation  $T$  in  $R$

defined by  $\mathbf{T}(a) = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$  and  $\mathbf{T}(b) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ . The commutator of  $\{\mathbf{T}(a), \mathbf{T}(b)\}$  in  $(R)_4$  is seen to be  $R(\zeta)$ , where  $\zeta = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$  is a primitive

fourth root of 1. Hence,  $RG$  contains  $(R(\zeta))_2$  as a simple component, and so  $U(RG)$  is not solvable.

**Theorem 2.** *Assume that  $C$  is of prime characteristic  $p$  and  $G$  contains a normal Sylow  $p$ -subgroup  $P$ . Then,  $U(RG)$  is solvable if and only if there holds one of the following :*

- (i)  $R=C$  and  $G$  is a semi-direct product of  $P$  and an abelian group.
- (ii)  $R=(\text{GF}(2))_2$  and  $G=P$ .
- (iii)  $R=(\text{GF}(3))_2$  and  $G$  is a semi-direct product of  $P$  and an abelian group of exponent 2.
- (iv)  $R=\text{GF}(3)$  and  $G$  is a semi-direct product of  $P$  and  $G_1 \times G_2$ , where  $G_1$  is an abelian group of exponent 2 and  $G_2$  is a non-abelian indecomposable 2-group such that  $U(RG_2)$  is solvable.

*Proof.* We consider the ring epimorphism  $\beta: RG \rightarrow RG^*$  defined by  $\sum_{\sigma \in G} a_\sigma \sigma \mapsto \sum_{\sigma \in G} a_\sigma \sigma^*$ , where  $G^* = G/P$  and  $\sigma^*$  is the residue class of  $\sigma$  modulo  $P$ . Then, by [2; Theorem 2],  $\text{Ker } \beta$  coincides with  $J(RG)$ . Hence,  $U(RG)$  is solvable if and only if so is  $U(RG^*)$ . Now, our assertion will be immediate by Theorem 1 and Schur-Zassenhaus theorem.

The following corollaries are only selections from Theorems 1 and 2.

**Corollary 1.** *Let  $C \neq \text{GF}(3)$ .*

- (a) *Assume that  $G \neq 1$  and  $g$  is not divisible by the characteristic of  $C$ . If  $U(RG)$  is solvable then it is nilpotent.*
- (b) *Assume that  $C$  is of prime characteristic  $p$  and  $G$  contains a proper normal Sylow  $p$ -subgroup  $P$ . If  $U(RG)$  is solvable then  $U(R \cdot G/P)$  is nilpotent, and conversely.*

**Corollary 2.** *Let  $C = \text{GF}(3)$ .*

- (a) *Assume that  $G$  is not a 2-group and  $g$  is not divisible by 3. If  $U(RG)$  is solvable then it is nilpotent.*

(b) Assume that  $G$  contains a normal Sylow 3-subgroup  $P$  such that  $G/P$  is not a 2-group. If  $U(RG)$  is solvable then  $U(R \cdot G/P)$  is nilpotent, and conversely.

**Corollary 3.** Let  $R$  be of prime characteristic  $p$ , and  $G$  a  $p$ -group. Then,  $U(RG)$  is solvable if and only if  $R=C$  or  $(\text{GF}(2))_2$  or  $(\text{GF}(3))_2$ .

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