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Some remarks on the bimodule structure of Galois extensions

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SOME REMARKS ON THE BIMODULE STRUCTURE OF GALOIS EXTENSIONS

Dedicated to Professor MASARU OSIMA on the occasion of his sixtieth birthday

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Throughout the present paper, A will represent an (Artinian) simple ring (with 1), B a unital simple subring of A such that the centralizer $V=V_A(B)$ of B in A is simple as well, and $\mathfrak{G}$ the group of B-ring automorphisms in A, and we use the following notations : $H=V_A(V)$, $C=V_A(A)$, $Z=V_B(B)$ and $C_0=V_V(V)=V\cap H$. Furthermore, we set $\mathfrak{R}^*=\mathrm{Hom}({}_BA_B, {}_BA_B)$, $\mathfrak{R}=\mathfrak{G}V_R$ and $\mathfrak{S}=V_L\cdot V_R$, which is isomorphic to $V^\circ\otimes_C V$, V° the opposite of V. Finally, as to notations and terminologies used without mention, we follow [2].

Recently, in his paper [1], S. Elliger obtained the following :

Theorem 1. *If A is finite Galois over B, then the following conditions are equivalent* :

(1) ${}_BA_B$ *is completely reducible.*

(2) $\mathfrak{R}$ *is semisimple.*

(3) C_0/C *is separable and* $\sum_{\eta\in\mathfrak{G}(H/B)}c\eta=1$ *for some* $c\in C_0$.

Theorem 2. *If A is finite Galois over B, then the following conditions are equivalent* :

(1) ${}_BA_B$ *is local.*

(2) $\mathfrak{R}$ *is a local ring.*

(3) i) $V=Z$ *and is purely inseparable over* C ;

ii) *either* A/B *is inner Galois or* $\mathfrak{G}(H/B)$ *is a p-group and* char $A=p$.

If A/B is finite outer Galois then there holds $\mathrm{Hom}({}_BA_B, {}_BB_B)=C_R\cdot\sum_{\sigma\in\mathfrak{G}}\sigma$ (see, for instance [2; Prop. 9.6]), whence we can easily see that ${}_BB_B<\oplus{}_BA_B$ (B is a direct summand of A as B-B-module) if and only if $\sum_{\sigma\in\mathfrak{G}}c\sigma=1$ for some $c\in C$. The last remark enables us to restate Theorem 1 as follows :

Theorem 1′. *If A is finite Galois over B, then the following con-*

ditions are equivalent:

(1) ${}_BA_B$ *is completely reducible.*

(2) ${}_HA_H$ *and* ${}_BH_B$ *are completely reducible.*

(3) C_0/C *is separable and* ${}_BB_B < \oplus {}_BH_B$.

Now, let M be a module with an operator domain Ω. If the ring of Ω-endomorphisms of M is a local ring (completely primary ring), M is said to be *completely indecomposable.* In fact, a completely indecomposable module is indecomposable, and an indecomposable module satisfying both chain conditions is completely indecomposable. In case A/B is finite Galois, it is known that ${}_BA_B$ is indecomposable if and only if it is local ([1; Th. 1]). Accordingly, Theorem 2 may be restated as follows:

Theorem 2′. *If A is finite Galois over B, then the following conditions are equivalent*:

(1) ${}_BA_B$ *is indecomposable.*

(2) ${}_HA_H$ *and* ${}_BH_B$ *are indecomposable.*

(3) i) $V=Z$ *and is purely inseparable over* C;
  ii) *either* A/B *is inner Galois or* $\mathfrak{G}(H/B)$ *is a p-group and* char $A=p$.

In this paper, the main theme of our discussion will concern the bimodule structure of Galois extensions of simple rings. One of the purposes of this paper is to extend Theorems 1′ and 2′ to a somewhat wider class of Galois extensions.

The next will be found in [2; Ths. 21.1 and 19.3], and will be used occasionally in our subsequent study.

Lemma 1. *Let A be Galois and left algebraic over B, and $[V:C] < \infty$. Then, A/B is $\mathfrak{G}$-locally Galois and A/A' is Galois for every regular intermediate ring A' of A/B.*

In case A/B is left locally finite, A/B is h-Galois if and only if A/B is Galois and A is $B \cdot V \cdot A$-irreducible ([2; Cor. 17.12]). Combining this with [2; Lemma 5.8], we readily obtain the following:

Lemma 2. *Let A be h-Galois and left locally finite over B. If T is a finitely generated B-B-submodule of A (which is necessarily left B-free) then $T|\mathfrak{R}$ contains a free V_R-basis that forms at the same time a free A_R-basis of $T|\mathfrak{G}A_R$, $[T|\mathfrak{R}: V_R] = [T|\mathfrak{G}A_R : A_R] = [T:B]_L < \infty$ and $T|\mathfrak{R} = \mathrm{Hom}({}_BT_F, {}_BA_B)$, which implies that $\mathfrak{R}$ is dense in $\mathfrak{R}^*$ (with respect to finite topology).*

First, we shall prove that Theorem 1′ is still valid for a left algebraic Galois extension A/B with $[V:C]<\infty$.

Theorem 3. *Let A be Galois and left algebraic over B. If $[V:C]<\infty$, then the following conditions are equivalent*:

(1) ${}_BA_B$ *is completely reducible.*

(2) ${}_HA_H$ *and* ${}_BH_B$ *are completely reducible.*

(3) C_0/C *is separable and* ${}_BB_B<\oplus_BH_B$ (*or* ${}_BB_B<\oplus_BA_B$).

Proof. Since $[V:C]<\infty$, $\mathfrak{S}$ is semisimple if and only if C_0/C is separable. Moreover, by Th. 1, ${}_HA_H$ is completely reducible if and only if $\mathfrak{S}$ is semisimple.

(2)$\Leftrightarrow$(3): By the above remark, it remains only to prove that ${}_BH_B$ is completely reducible if (and only if) ${}_BB_B<\oplus_BH_B$. If F is an arbitrary finite subset of H then we can find an intermediate ring H' of $H/B[F]$ which is finite (outer) Galois over B (Lemma 1). Since ${}_BB_B<\oplus_BH_B$ yields ${}_BB_B<\oplus_BH'_B$, ${}_BH'_B$ is completely reducible by Th. 1′, which implies the complete reducibility of ${}_BH_B$.

(1)$\Rightarrow$(3): Evidently, ${}_BH_B$ is completely reducible. Suppose $\mathfrak{S}$ has non-zero (nilpotent) radical $\mathfrak{N}$. Noting that $\mathfrak{N}$ is dense in $\mathfrak{N}^*$ (Lemma 2), we can easily see that $\mathfrak{N}^*\mathfrak{N}$ is nilpotent. But, this contradicts the fact that $\mathfrak{N}^*$ is regular in von Neumann's sense. Hence, $\mathfrak{S}$ is semisimple.

(3)$\Rightarrow$(1): As is well-known, there exists an element u with $C_0=C[u]$. Given a finite subset F of A, Lemma 1 enables us to find $A'\in\mathfrak{R}_{l.f}/B[F,u]$ such that A'/B is Galois, $V_A(A')=C$ and $V=V'.C$ where $V'=V_{A'}(B)$. Then, $H'=V_{A'}(V')=V_{A'}(V'.C)=V_{A'}(V)\subseteq H$, $C_0'=V'\cap H'=A'\cap C_0\ni u$ and $C'=V_{A'}(A')=A'\cap C$. Setting $\mathfrak{G}'=\mathfrak{G}(A')$, we have $J(C_0|\mathfrak{G}')=C_0'$ and $J(C|\mathfrak{G}')=C'$ (Lemma 1). If $\sum_0^{t-1}c_iu^i$ is in C_0' ($c_i\in C$, $t=[C_0:C]$), then for every $\sigma\in\mathfrak{G}'$ there holds $\sum(c_i\sigma-c_i)u^i=(\sum c_iu^i)\sigma-\sum c_iu^i=0$, which means $c_i\in C'$ and hence $C_0'=C'[u]$. Needless to say, C_0' is separable over C'. Furthermore, ${}_BH'_B(\subseteq{}_BH_B)$ is completely reducible. From those, ${}_BA'_B$ is completely reducible by Th. 1′. Now, (1) is an easy consequence of the last.

Corollary 1. *Let A be a central simple algebra of finite rank over C which is Galois and left algebraic over B. If C_0 is separable over C then ${}_BA_B$ is completely reducible, and conversely.*

Proof. Since $[B:Z]<\infty$ by [2; Cor. 7.11], $B\cdot C_0=B\otimes_ZC_0$ is a simple intermediaite ring of H/C. Hence, H coincides with $B\cdot C_0$ and ${}_BH_B$

is homogeneously completely reducible. Now, our assertion is clear by Th. 3.

Corollary 2. *Let A be outer Galois and left algebraic over B. If ${}_BA_B$ is completely reducible then so is ${}_{B'}A_{B'}$ for every intermediate ring B' of A/B.*

Proof. We shall divide the proof into two steps. We consider first the case where A is finite over B. By Th. 1, there exists an element $c\in C$ such that $\sum_{\sigma\in\mathfrak{G}}c\sigma=1$. Let $\mathfrak{G}'=\mathfrak{G}(A/B')$, and $\{\sigma_1, \cdots, \sigma_k\}$ a left representative system of $\mathfrak{G}$ modulo $\mathfrak{G}'$. Then, $c'=\sum_1^k c\sigma_i$ is in C and $\sum_{\sigma'\in\mathfrak{G}'}c'\sigma'=1$. Hence, ${}_{B'}A_{B'}$ is completely reducible again by Th. 1. Next, we shall consider the general case. If F is an arbitrary finite subset of A then there exists a $\mathfrak{G}$-invariant $N\in\mathcal{R}_{l.f}/F$ by Lemma 1 and [2; Prop. 16.4]. By the first step, there exists then an element $c\in V_N(N)$ such that $\sum_{\sigma\in\mathfrak{G}(N/N')}c\sigma=1$, where $N'=N\cap B'$. Moreover, by [2; Th. 8.1], $B'[N]$ is outer Galois and finite over B' and $\mathfrak{G}(B'[N]/B')$ is isomorphic to $\mathfrak{G}(N/N')$ by the contraction map. Accordingly, it follows $\sum_{\sigma\in\mathfrak{G}(B'[N]/B')}c\sigma=\sum_{\sigma\in\mathfrak{G}(N/N')}c\sigma=1$. Noting here that c is evidently contained in the center of $B'[N]$, we see that ${}_{B'}B'[N]_{B'}$ is completely reducible, proving the complete reducibility of ${}_{B'}A_{B'}$.

Assume here that A/B is outer Galois and left algebraic. Then, $\mathfrak{G}$ is l. f. d. by [2; Prop. 16.4] and $\mathfrak{G}$ is the pro-finite group $\varprojlim \mathfrak{G}_\alpha$ where $\mathfrak{G}_\alpha=\mathfrak{G}(A_\alpha/B)=A_\alpha|\mathfrak{G}$ and A_α ranges over all the $\mathfrak{G}$-invariant (simple) intermediate rings of A/B finite over B. (See [2; p. 115].) Now, under the above notations, we shall prove the following:

Lemma 3. *If A is outer Galois and left algebraic over a proper simple subring B then the following conditions are equivalent*:

(1) *${}_BA_B$ is completely indecomposable, namely, $\mathfrak{R}^*$ is a local ring.*

(2) *$C=Z$,* char $A=p$, *and $\mathfrak{G}$ is a pro-p-group, namely, every $\mathfrak{G}_\alpha\neq 1$ is a p-group.*

Proof. (1)$\Rightarrow$(2): C/Z is Galois with $C|\mathfrak{G}$ as a Galois group. It follows therefore $(C|\mathfrak{G})C_R$ is dense in $\mathfrak{L}=\mathrm{Hom}({}_ZC, {}_ZC)$. Recalling that $\mathfrak{R}=\mathfrak{G}C_R$ is dense in $\mathfrak{R}^*$ (Lemma 2), we see that $C|\mathfrak{R}^*$ is dense in $\mathfrak{L}$. Suppose $[C:Z]>1$, and consider a Z-subspace T of C with $[T:Z]=2$. We set $\mathfrak{T}=\{\tau\in C|\mathfrak{R}^*|T\tau\subseteqq T\}$. We shall prove that $\mathfrak{T}$ is a local ring. In fact, if $\tau\in\mathfrak{T}$ is a unit in the local ring $C|\mathfrak{R}^*$ then $T\tau=T$, whence we obtain $\tau^{-1}\in\mathfrak{T}$. We can see therefore that if τ and τ' are non-units of $\mathfrak{T}$ then $\tau+\tau'$ is a non-unit, namely, $\mathfrak{T}$ is a local ring. But, $(Z)_2$ is a homomorphic image of $\mathfrak{T}$. This contradiction implies $C=Z$. Hence, $\mathrm{Hom}({}_BA_{\alpha B}, {}_BA_{\alpha B})=A_\alpha|\mathfrak{R}^*$ is the group ring $\mathfrak{G}_\alpha Z_R$. Accordingly, by [2; Lemma 13.4], char $B=p$ and $\mathfrak{G}_\alpha$ is a p-group, provided $A_\alpha\neq B$.

(2)$\Rightarrow$(1): Again by [2; Lemma 13.4], $\mathrm{Hom}({}_BA_{\alpha B}, {}_BA_{\alpha B})=A_\alpha|\mathfrak{R}^*=\mathfrak{G}_\alpha Z_R$ is a local ring. If ϕ and ψ are non-units of $\mathfrak{R}^*$ then there exists some A_β such that $A_\beta|\phi$ and $A_\beta|\psi$ are non-units. Hence, $A_\beta|\phi+\psi$ is a non-unit, which implies that $\phi+\psi$ is a non-unit. Thus, we have shown that $\mathfrak{R}^*$ is a local ring.

Theorem 4. *Let A be Galois and left algebraic over B. If* $[V:C]<\infty$, *then the following conditions are equivalent*:

(1) *${}_BA_B$ is completely indecomposable.*

(2) *${}_HA_H$ and ${}_BH_B$ are completely indecomposable.*

(3) i) *$V=Z$ and is purely inseparable over C*;

ii) *either A/B is inner Galois or $\mathfrak{G}(H/B)$ is a pro-p-group and* char $A=p$.

Proof. If $\mathfrak{S}$ is a local ring then so is $C_0\otimes_C C_0$, namely, C_0/C is purely inseparable. If $[V:C_0]=m$ then $(C_0)_m\cong V^\circ\otimes_{C_0}V$ is a homomorphic image of the local ring $\mathfrak{S}$. Hence, it follows $V=C_0$. Combining this with Lemma 3, we readily obtain the equivalence (2)$\Leftrightarrow$(3).

(1)$\Rightarrow$(3): Evidently, $\mathfrak{S}\cong V^\circ\otimes_C V$ is an Artinian subring of the local ring $\mathfrak{R}^*$. Hence, $\mathfrak{S}$ is a local ring and, as was noted above, $V=C_0$ and is purely inseparable over C. Hence, $C_0|\mathfrak{R}^*$ is dense in $\mathfrak{L}=\mathrm{Hom}({}_ZC_0, {}_ZC_0)$. Now, by making use of the same argument as in the proof of Lemma 3 (1)$\Rightarrow$(2), we can easily see that $C_0=Z$ and if $H\neq B$ then char $B=p$ and $\mathfrak{G}(H/B)$ is a pro-p-group.

(3)$\Rightarrow$(1): By Lemmas 2 and 3, $\mathfrak{R}=\mathfrak{G}Z_R$ is dense in $\mathfrak{R}^*$ and $H|\mathfrak{R}$ is dense in the local ring $\mathfrak{B}^*=\mathrm{Hom}({}_BH_B, {}_BH_B)$. Accordingly, if A' is a $\mathfrak{G}$-invariant simple intermediate ring of A/B left finite over B then for every $\phi\in\mathfrak{R}^*$ there holds $A'\phi\subseteq A'$. From this, noting that $\mathfrak{G}$ is l. f. d. by [2; Prop. 16.4], one will easily see that if ϕ and ψ are non-units of $\mathfrak{R}^*$ then there exists a $\mathfrak{G}$-invariant simple intermediate ring A'' of A/B left finite over B such that $V_A(A'')=C$ and both $A''|\phi$ and $A''|\psi$ are non-units of $A''|\mathfrak{R}^*$. Evidently, A''/B is finite Galois and $V_{A''}(B)=Z$ is purely inseparable over $V_{A''}(A'')=C$. Moreover, setting $H''=V_{A''}(Z)$, $\mathrm{Hom}({}_BH''_B, {}_BH''_B)=H''|\mathfrak{B}^*$ is a local ring. From those, we see that Hom

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$({}_BA''_B, {}_BA''_B)=A''|\mathfrak{R}^*$ is a local ring (Th. 2′). Hence, $A''|\phi+\psi_r$ is a non-unit, which implies that $\phi+\psi_r$ is a non-unit. Thus, we have shown that ${}_BA_B$ is completely indecomposable.

Corollary 3. *Let A be a central simple algebra of finite rank over C which is Galois and left algebraic over B. If ${}_BA_B$ is completely indecomposable then A/B is (finite) inner Galois and Z is purely inseparable over C, and conversely.*

Proof. As was noted in the proof of Cor. 1, H coincides with $B\cdot C_0$. Accordingly, $H=B\cdot C_0=B\cdot Z=B$ and Z/C is purely inseparable by Th. 4.

Remark 1. The following example shows that Cor. 2 is not always true even for finite Galois extensions: Let P/Φ be a two dimensional purely inseparable field extension. If we set $A=(\Phi)_2$ and $B=\Phi$ then A/B is finite inner Galois and there exists an intermediate ring B' of A/B which is B-isomorphic to P. By Th. 1′, ${}_{B'}A_{B'}$ is not completely reducible but ${}_BA_B$ is obviously completely reducible.

Remark 2. If ${}_BA_B$ is completely indecomposable then so is ${}_{B'}A_{B'}$ for every intermediate ring B' of A/B. In fact, $\mathrm{Hom}({}_{B'}A_{B'}, {}_{B'}A_{B'})=V_{\mathfrak{R}^*}(B'_L\cdot B'_R)$ is a local ring.

Remark 3. If H is simple (for instance, if $[V:C]<\infty$), then the following conditions are equivalent:

(1) ${}_BA_B$ is homogeneously completely reducible, namely, $A=B\cdot V$.

(2) ${}_HA_H$ and ${}_BH_B$ are homogeneously completely reducible.

In fact, this is an easy consequence of $B\cdot V=B\otimes_Z V$ and corresponds to the equivalence between (1) and (2) in Th. 3 or Th. 4. Moreover, we obtain the following:

(a) *Let A be a central simple algebra of finite rank over C. If $C_0=C$ then ${}_BA_B$ is homogeneously completely reducible, and conversely.*

(b) *Let $[A:B]_L<\infty$. If Z is a subfield of C and $[C:Z]=[H:B]_L$ then ${}_BA_B$ is homogeneously completely reducible, and conversely.*

Proof. (a) Evidently, H is simple and $H\cdot V=H\otimes_C V$ is a simple intermediate ring of A/C. Accordingly, A coincides with $H\cdot V$. On the other hand, as was noted in the proof of Cor. 1, H coincides with $B\cdot C$, and hence $A=B\cdot V$.

(b) By [2; Prop. 5.4], $[V:C]\leqq[A:B]_L$ and H is simple. Now, our assertion is an easy consequence of $B\cdot V=B\otimes_Z V$ and $[V:C]=[A:H]$.

By the way, one may remark here that if $[V:C]<\infty$ then $[A:C]<\infty$ and $[B:Z]<\infty$ are equivalent by [2; Cors. 4.9 and 7.11].

Remark 4. Let $\mathfrak{G}'$ be an F-group in A with $B=J(\mathfrak{G}')$. Assume that $\mathfrak{G}'$ is a p-group and char $A=p$.

(a) *If* ${}_BA_B$ *is completely reducible then* A/B *is outer Galois and* ${}_BA_B$ *is homogeneously completely reducible.*

(b) ${}_BA_B$ *is local if and only if* B *contains* C.

Proof. By [2; Lemma 10.4], V coincides with $Z\cdot C$ and is purely inseparable over C. Hence, noting that $\mathfrak{G}(H/B)\cong\mathfrak{G}/\widetilde{V}\cong\mathfrak{G}'/\mathfrak{G}'\cap\widetilde{V}$, (b) is an easy consequence of Th. 2. Now, we shall prove (a). Since $Z\cdot C/C$ is separable as well (Th. 3), $Z\cdot C$ coincides with C, namely, A/B is outer Galois. It follows therefore $\mathfrak{G}'=\mathfrak{G}$. By Th. 1, there exists an element $c\in C$ such that $\sum_{\sigma\in\mathfrak{G}}c\sigma=1$. Hence, we obtain $A=B\cdot C$ by [2; Cor. 13.9].

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