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ON THE BIMODULE STRUCTURE OF GALOIS EXTENSIONS

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Throughout the present note, A will represent an (Artinian) simple ring which is finite Galois over a simple subring B . Then, it is known that A is $B \cdot V_A(B)$ - A -irreducible. (See [4]. As to terminologies used without mention, we follow [4].) We use the following notations: $C = V_A(A)$, $Z = V_B(B)$, $V = V_A(B)$, $H = V_A(V)$, $C_0 = V_V(V) = V \cap H$, G = the Galois group of A/B , $\{\sigma_1 = 1, \sigma_2, \dots, \sigma_n\}$ is a (fixed) representative system of G modulo the normal subgroup I consisting of all inner automorphisms, and $\bar{G} = G/I$, which may and will be regarded as the Galois group of H/B . We set further $S = V_l \cdot V_r$ and $R = \text{Hom}({}_B A_B, {}_B A_B)$, where V_l (resp. V_r) the left (resp. right) multiplication of V . To be easily seen, $R = GV_r = \bigoplus_{i=1}^n \sigma_i S$ and $S \cong V^\circ \otimes_C V$ is a Frobenius ring, where V° is the opposite of V . (See, for instance [3; Lemma 3].)

In this note, the main theme of our discussion will concern the bimodule structure of ${}_B A_B$. We shall prove first that ${}_B A_B$ is a direct sum of local submodules, where a module ${}_B M_B$ is said to be *local* if it contains one and only one maximal submodule (Theorem 1). Next, we shall explain when ${}_B A_B$ is completely reducible (Theorem 2) or local (Theorem 3).

The next lemma will play an essential role in our subsequent study.

Lemma 1. *If T is a B - B -submodule of A then the restriction $T|R$ of R to T contains a free V_r -basis that forms at the same time a free A_r -basis of $T|GA_r$, $[T|R:V_r] = [T|GA_r:A_r] = [T:B]_l$ and $T|R = \text{Hom}({}_B T_B, {}_B A_B)$.*

Proof. As is well-known, T is left (resp. right) B -free. Hence, the lemma is contained in [4; Lemma 5.8].

Theorem 1. *R is a Frobenius ring and ${}_B A_B$ is a direct sum of local modules.*

Proof. To be easily seen, the map h defined by $\sum_{i=1}^n \sigma_i s_i \mapsto s_1$ is a Frobenius homomorphism of the ring extension R/S , namely, R/S is a free Frobenius extension. Since S is a Frobenius ring, so is R by [1;

Satz 10]. Next, we shall prove the latter part. To our end, it is enough to show that if e is a primitive idempotent of R then ${}_BAe_B$ is local. Let T_1 and T_2 be arbitrary proper B - B -submodules of Ae . Noting that $[T_i|R:V_r] = [T_i:B]_i \leq [Ae:B]_i = [Ae|R:V_r]$ (Lemma 1), the kernel of the restriction map $h_i: Ae|R \rightarrow T_i|R$ is non-zero. Since R is a Frobenius ring, $\ker h_1 \cap \ker h_2$ contains a submodule isomorphic to the unique minimal right subideal of eR . Hence, the kernel of the restriction map $Ae|R \rightarrow T_1+T_2|R$ is non-zero, which implies $T_1+T_2 \neq Ae$ and that ${}_BAe_B$ is local.

Theorem 2. *The following conditions are equivalent :*

- (1) ${}_BA_B$ is completely reducible.
- (2) R is semisimple.
- (3) a) C_0/C is separable ;
b) $\sum_{i=1}^n c\sigma_i = 1$ for some $c \in C_0$.

Proof. (1) $\Rightarrow$ (2): This is evident by Lemma 1.

(2) $\Rightarrow$ (1): It suffices to prove that if $eR(e^2=e)$ is a minimal right ideal then ${}_BAe_B$ is irreducible. Let T' be an arbitrary non-zero B - B -submodule of Ae . Noting that $Ae|R$ is R -irreducible, we obtain $[T':B]_i = [T'|R:V_r] = [Ae|R:V_r] = [Ae:B]_i$ (Lemma 1), namely, $T' = Ae$.

(2) $\Rightarrow$ (3): Evidently, the semisimplicity of R implies the semisimplicity of S , equivalently, the separability of C_0/C . Since ${}_BA_B$ is then completely reducible, so is ${}_BH_B$. Accordingly, $\text{Hom}({}_BH_B, {}_BH_B) = \overline{GC}_0$ is semisimple. Hence, C_0 is a direct summand of H as $\overline{GC}_0$ -module. Now, noting that H contains an element a with $1 = \sum_{\bar{\sigma} \in \bar{G}} a\bar{\sigma} = \sum_{i=1}^n a\sigma_i$, we readily obtain (iii) b).

(3) $\Rightarrow$ (2): Let $\sum_{i=1}^n c\sigma_i = 1$ for some $c \in C_0$. To be easily verified, $\sum_{i=1}^n \sigma_i^{-1}c_r\sigma_i = 1$ and $\sum_{i=1}^n x\sigma_i^{-1}c_r \otimes \sigma_i = \sum_{i=1}^n \sigma_i^{-1}c_r \otimes \sigma_i x$ (in $R \otimes_S R$) for every $x \in R$. This means that R is a separable extension of the semisimple ring S . Then, R is semisimple by [2; Lemma 2.10 (1)].

Corollary 1. *If C_0 is separable over C and n is not divisible by char A , then ${}_BA_B$ is completely reducible.*

Theorem 3. *The following conditions are equivalent :*

- (1) ${}_BA_B$ is local.
- (2) R is a local ring.
- (3) a) $V = Z$ and is purely inseparable over C ;

b) *either A/B is inner Galois or $\bar{G}$ is a p -group and $\text{char } A = p$.*

Proof. (1) $\Rightarrow$ (2): This is evident by Th. 1.

(2) $\Rightarrow$ (3): Since R is local, so is the subring S which is isomorphic to $V^\circ \otimes_c V$. Accordingly, $C_0 \otimes_c C_0$ is a local ring, namely, C_0/C is purely inseparable. If $[V : C_0] = m$ then $(C_0)_m (\cong V^\circ \otimes_{C_0} V)$ is a homomorphic image of the local ring S . Hence, it follows $V = C_0$. Further, C_0/Z is Galois with $C_0/\bar{G}$ as Galois group, and so if $[C_0 : Z] = t$ then $(Z)_t (\cong C_0/\bar{G}C_0)$ is homomorphic to R . It follows therefore $Z = C_0 = V$ and $H|R$ is isomorphic to the group ring of $\bar{G}$ over Z . Hence, if $n > 1$ then $\bar{G}$ is a p -group and $p = \text{char } Z$ by [4; Lemma 13.4].

(3) $\Rightarrow$ (2): Let ϕ be the ring homomorphism of the local ring $S = Z_1 \cdot Z_r$ onto Z given by $\sum z_{i1} \cdot z'_{ir} \mapsto \sum z_i z'_i$. Then, the kernel of ϕ is the radical J of S . Now, one will easily see that the kernel of the restriction map $R \rightarrow H|R$ is $\bigoplus_{i=1}^n \sigma_i J$ and nilpotent. Since $H|R = \bar{G}Z_r$ is a local ring again by [4; Lemma 13.4], we can easily see that R is a local ring.

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