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An equivalence relation in topology

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AN EQUIVALENCE RELATION IN TOPOLOGY

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1. Equivalent sets. Introduction

It seems reasonable to define equality or equivalence of sets in a topological space X in some way which involves the topology. After some experimenting, we came upon the following:

Definition 1.1. In a space X, A is *equivalent* to B (written $A \equiv B$) iff for each open set O, $A \subseteq O$ iff $B \subseteq O$.

We shall make frequent use of

Lemma 1.2. *In a space X, $A \equiv B$ iff $a \in A$ implies that $c(a) \cap B \neq \emptyset$ and $b \in B$ implies that $c(b) \cap A \neq \emptyset$, c denoting the closure operator.*

Proof. Let $A \equiv B$ and take $a \in A$. Then $A \nsubseteq Cc(a)$ and $Cc(a)$ is open, C denoting the complement operator. Thus $B \nsubseteq Cc(a)$ and hence $B \cap c(a) \neq \emptyset$.

Conversely, suppose that $A \not\equiv B$. We may assume that there exists an open set O such that $A \subseteq O$ and $B \nsubseteq O$; take $b \in B \cap CO$. Then $c(b) \subseteq CO \subseteq CA$ and hence $c(b) \cap A = \emptyset$.

Theorem 1.3. *If O and U are open in X, then $O \equiv U$ iff $O = U$.*

We shall often refer to

Example 1.4. Let $X = \{a, b\}$ with open sets $\emptyset$, $\{a\}$, X. Then $\{b\} \equiv X$ and both sets are closed, but equality fails (see Theorem 1.6). Note also that a set equivalent to an open set need not be open.

Definition 1.5. For each set $A \subseteq X$, let $A^* = \cap\{O : A \subseteq O$ and O is open$\}$.

Theorem 1.6. *In a space X, $A \equiv B$ iff $A^* = B^*$.*

Proof. Let $A \equiv B$ and take $a^* \in A^*$. If $a^* \notin B^*$, then $a^* \notin O$ for some open set which contains B. But then $A \subseteq O$ and hence $a^* \notin A^*$, a contradiction.

Conversely, let $A^* = B^*$ and suppose that $A \subseteq O$, O being open. Then $B \subseteq B^* = A^* \subseteq O$ and hence $B \subseteq O$. It follows then that $A \equiv B$.

Theorem 1.7. * *as defined in Definition* 1.5 *is a Kuratowski closure*

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operator.

Proof. $\emptyset^* = \emptyset$ and $A \subseteqq A^*$ are clear. If $x \notin (A \cup B)^*$, then $x \notin O$ for some open set such that $A \cup B \subseteqq O$. Then $x \notin A^* \cup B^*$. Conversely, if $x \notin A^* \cup B^*$, then $x \notin O$ for some open set such that $A \subseteqq O$ and $x \notin U$ for some open set such that $B \subseteqq U$. Thus $A \cup B \subseteqq O \cup U$ and $x \notin O \cup U$. Hence $x \notin (A \cup B)^*$. It remains to show that $A^{**} \subseteqq A^*$; suppose that $x \notin A^*$. Then there exists an open set O such that $x \notin O$, $A \subseteqq O$. Then $x \notin O$ and $A^* \subseteqq O$ and hence $x \notin A^{**}$.

Theorem 1.8. *In a space X, A^* is the largest set which is equivalent to A.*

Proof. Clearly, $A \equiv A^*$. Suppose then that $B \equiv A$. Then for each open set O such that $A \subseteqq O$, then $B \subseteqq O$. It follows then that $B \subseteqq A^*$.

In general, there is no smallest set which is equivalent to a given set. However, we have

Theorem 1.9. *In a space X, let A be closed and compact. There exists a smallest closed set B which is equivalent to A.*

Proof. Let $B = \cap \{A' : A'$ is closed and $A' \equiv A\}$. It suffices to show that $B \equiv A$. Since $B \subseteqq A$, it suffices to show that $A \subseteqq O$ if $B \subseteqq O$ and O is open. $B \subseteqq O$ implies that $\cap \{A'_i : 1 \leqq i \leqq n\} \subseteqq O$ and $A \equiv A'_1 \cap \cdots \cap A'_n$ (see Corollary 2.4). Thus $A \subseteqq O$.

Equivalence of sets is an absolute property as shown in

Theorem 1.10. *Let Y be a subspace of X and A, $B \subseteqq Y$. Then $A \equiv B$ (in Y) iff $A \equiv B$ (in X).*

Theorem 1.11. *Let $f : X \to Y$ be continuous and suppose that $A \equiv B$ in X. Then $f[A] \equiv f[B]$ in Y.*

Theorem 1.12. *In a space X, all nonempty closed sets are equivalent iff $O \neq X$, O open implies that O has no nonempty closed subsets.*

Proof. Suppose that $X \neq O$, O open and that $O \supseteqq E \neq \emptyset$, with E closed. Then CO and E are nonempty closed sets which are not equivalent. Conversely, suppose that $E \neq \emptyset \neq F$, E and F being closed and non equivalent sets. We may assume that $E \subseteqq O$ and $F \nsubseteq O$ for some open set O. Then $O \neq X$ and O has a nonempty closed subset.

Corollary 1.13. *Let $\mathcal{T}$ be a chain topology for X. Then $E \equiv F$ if E and F are nonempty closed sets.*

Proof. By Theorem 1.12, it suffices to show that O has no nonempty closed subset if O is open and $O \neq X$. If $O \supseteqq E \neq \emptyset$, E closed, then O and CE are non comparable open sets and $\mathcal{T}$ is not a chain topology, a contradiction.

The converse of Corollary 1.13 is false as shown in

Example 1.14. Let $X = \{a, b, c, d\}$ with open sets $\mathcal{T}$: $\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X$. Then $\mathcal{T}$ is not a chain topology for X, but all nonempty closed sets contain d and the only open set which contains d is X. Thus all nonempty closed sets are equivalent.

2. The algebra of equivalent sets

Theorem 2.1. *In a space X, let $A_\alpha \equiv B_\alpha$ for each $\alpha \in \triangle$. Then* (1) $\cup \{A_\alpha : \alpha \in \triangle\} \equiv \cup \{B_\alpha : \alpha \in \triangle\}$ *and* (2) *for each* $A \subseteqq X$, $A^* = \cup \{B : B \equiv A\}$.

We omit the easy proof.

If $A \equiv B$, it does not generally follow that $A \cap C \equiv B \cap C$. However, we have

Theorem 2.2. *In a space X, let $A \equiv B$ and let E be a closed subset of X. Then $A \cap E \equiv B \cap E$.*

Proof. Let $A \cap E \subseteqq O$, O being an open set. Then $A \subseteqq O \cup CE$ and hence $B \subseteqq O \cup CE$. Thus $B \cap E \subseteqq O$.

Theorem 2.3. *In a space X, let $A \equiv E$ and $A \equiv F$, E being closed. Then $A \equiv E \cap F$.*

Proof. If $A \subseteqq O$, O open, then $E \subseteqq O$ and hence $E \cap F \subseteqq O$. Conversely, let $E \cap F \subseteqq O$. Then $F \subseteqq O \cup CE$ and since $F \equiv E$, we have $E \subseteqq O \cup CE$. Then $E \subseteqq O$ implies that $A \subseteqq O$.

Corollary 2.4. *In a space X, let $A \equiv E_i$, $i = 1, \cdots, n$ where each E_i is closed. Then $A \equiv E_1 \cap \cdots \cap E_n$.*

Theorem 2.5. *Let $X = \times \{X_\alpha : \alpha \in \triangle\}$ and suppose that $A_\alpha \neq \emptyset \neq B_\alpha$ for each $\alpha \in \triangle$. Then $A_\alpha \equiv B_\alpha$ for each $\alpha \in \triangle$ iff $\times \{A_\alpha : \alpha \in \triangle\} \equiv \times \{B_\alpha : \alpha \in \triangle\}$.*

Proof. If $\times \{A_\alpha : \alpha \in \triangle\} \equiv \times \{B_\alpha : \alpha \in \triangle\}$, then $A_\alpha \equiv B_\alpha$ for each $\alpha \in \triangle$ by Theorem 1.11. Conversely, let $A_\alpha \equiv B_\alpha$ for each $\alpha \in \triangle$ and suppose that $x \in \times \{A_\alpha : \alpha \in \triangle\}$. Then $x(\alpha) \in A_\alpha$ for all $\alpha \in \triangle$ and hence

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$c(x(\alpha)) \cap B_\alpha \neq \emptyset$ by Lemma 1.2. It follows that $c(x) \cap \times \{B_\alpha : \alpha \in \triangle\} \neq \emptyset$.

3. Separation. R_0, T_0, T_1 spaces

Definition 3.1. A space X is called an R_0-*space* iff $x \in O$, O open implies that $c(x) \subseteq O$.

Theorem 3.2. *A space X is an R_0-space iff $c(x) \subseteq \{x\}^*$ for each $x \in X$* (see Definition 1.5).

Proof. If X is an R_0-space, then $x \in O$, O open implies that $c(x) \subseteq O$ and hence $c(x) \subseteq \cap \{O : x \in O,\ O \text{ open}\} = \{x\}^*$. Conversely, let $x \in O$, O open. Then $c\{x) \subseteq \{x\}^* \subseteq O$ and hence X is an R_0-space.

Theorem 3.3. *A space X is a T_0-space iff $x \neq y$ implies that $\{x\}^* \neq \{y\}^*$.*

Proof. Let X be a T_0-space and suppose that $x \neq y$. We may assume that $x \in O$, O open and $y \notin O$. Then $y \notin \{x\}^*$ and hence $\{y\}^* \neq \{x\}^*$. Conversely, suppose that $x \neq y$ implies that $\{x\}^* \neq \{y\}^*$. Let $x \neq y$ and assume that $\{x\}^* \nsubseteq \{y\}^*$; take $z \in \{x\}^*$ and $z \notin \{y\}^*$. There exists then an open set O containing y such that $z \notin O$. Then $x \notin O$a nd X is a T_0-space.

Theorem 3.4. *X is a T_1-space iff equivalence and equality coincide.*

Proof. Let X be a T_1-space and suppose that $A \equiv B$, but $A \nsubseteq B$. Let $a \in A$, $a \notin B$; then $B \subseteq \mathcal{C}\{a\}$, $\{a\}\mathcal{C}$ is an open set, but $A \nsubseteq \mathcal{C}\{a\}$, a contradiction.

Conversely, suppose that equality and equivalence coincide, but that $\{x\} \neq c(x)$ for some $x \in X$. Then $c(x) - \{x\} \neq c(x)$ and hence $c(x) - \{x\} \not\equiv c(x)$. There exists then an open set O such that $c(x) - \{x\} \subseteq O$, but $c(x) \nsubseteq O$ and hence $x \in \mathcal{C}O$. It follows then that $c(x) \subseteq \mathcal{C}O$, a contradiction.

4. Compactness

Theorem 4.1. *In a space X, let $A \equiv B$ and suppose that A is compact (Lindelöf, countably compact). Then B is compact (Lindelöf, countably compact).*

Theorem 4.2. *In a space X, let $A \equiv B$ and suppose that A is* *sequentially compact. Then B is sequentially compact.*

Proof. Let $\{b_i\}$ be a sequence in B. By Lemma 1.2, $c(b_i)\cap A\neq\emptyset$ for each i; take $a_i\in c(b_i)\cap A$ for each i. Then there exists an $a\in A$ and a subsequence $\{a_{n_i}\}$ which converges to a. Let $b\in c(a)\cap B$. Then $\lim b_{n_i}=b$; if $b\in O$, O open, then $a\in O$ and hence $a_{n_i}\in O$ for all $i\geqq N$. Then $b_{n_i}\in O$ for all $i\geqq N$.

Theorem 4.3. *In a space X, let A be locally compact and CA compact. If $A\equiv B$ and B is closed, then B is locally compact* (see Theorem 10.3).

Proof. Let $b\in B$. By Lemma 1.2, $c(b)\cap A\neq\emptyset$; take $a\in c(b)\cap A$. Then $a\in O\cap A\subseteqq M\subseteqq A$ for some open set O and some compact set M. Then $b\in O\cap B\subseteqq B\cap(M\cup CA)$ and $B\cap(M\cup CA)$ is a compact subset of B.

5. Uniform spaces

Theorem 5.1. *Let $(X, \mathcal{U})$ be a uniform space and $A\equiv B$ in X. If A is complete, then B is complete.*

Proof. Let $S: D\to B$ be a Cauchy net. Then by Lemma 1.2, $c(S(d))\cap A\neq\emptyset$ for each $d\in D$; let $a_d\in c(S(d))\cap A$ for each $d\in D$ and let $T: D\to A$ via $T(d)=a_d$. Then $T: D\to A$ is a Cauchy net. To see this, let $U\in\mathcal{U}$, U closed. Then $(S(d'), S(d''))\in U$ for all $d', d''\geqq d^*$ and hence $(a_{d'}, a_{d''})\in c(S(d'), S(d'')\subseteqq U$ for all $d', d''\geqq d^*$. Since A is complete, there exists an $a\in A$ such that $\lim T=a$. Let $b\in c(a)\cap B$. Then $\lim S=b$; for if $b\in O$, O open, then $a\in O$ and hence $T(d)\in O$ for all $d\geqq M$. It follows that $S(d)\in O$ for all $d\geqq M$.

Theorem 5.2. *Let $(X, \mathcal{U})$ be a uniform space and $A\equiv B$ in X. If A is totally bounded, so is B.*

Proof. Let $U\in\mathcal{U}$, U open. Then there exist $a_i\in A$ such that $A\subseteqq U[a_1]\cup\cdots\cup U[a_n]$. By Lemma 1.2, $c(a_i)\cap B\neq\emptyset$; take $b_i\in c(a_i)\cap B$. Then $B\subseteqq U[a_1]\cup\cdots\cup U[a_n]$ since $U[a_i]$ is open. $U[a_i]\subseteqq U[b_i]$ implies that $B\subseteqq U[b_1]\cup\cdots\cup U[b_n]$.

6. R_0-spaces. Introduction

Theorem 6.1. *Let X be an R_0-space* (see Example 3.1). *If $A\equiv B$*

and A *and* B *are closed, then* $A=B$ (see Example 1.4).

Proof. Let $a\in A$ and suppose that $a\notin B$. Then $a\in CB$ and hence $c(a)\subseteq CB$. Thus $c(a)\cap B=\emptyset$, contrary to Lemma 1.2.

Theorem 6.2. *Let* X *be an* R_0*-space and* $A\equiv B$ *in* X. *If* A *is dense, then* B *is dense.*

Proof. Let O be a nonempty open set. Then $A\cap O\neq\emptyset$; take $a\in A\cap O$. Then $c(a)\cap B\neq\emptyset$ by Lemma 1.2 and $c(a)\subseteq O$. Thus $O\cap B\supseteq c(a)\cap B\neq\emptyset$.

In Example 1.4, $\{b\}\equiv X$, but $\{b\}$ is not dense.

Theorem 6.3. *Let* X *be an* R_0*-space and* $A\equiv B$ *in* X. *If* O *is an open set, then* $A\cap O\equiv B\cap O$ (see Theorem 2.2).

Proof. Let $a\in A\cap O$. Then $c(a)\cap B\neq\emptyset$ by Lemma 1.2. But $c(a)\cap B\cap O=c(a)\cap B\neq\emptyset$. Using Lemma 1.2 again, $A\cap O\equiv B\cap O$.

In Example 1.4, let $O=\{a\}$. Then $\{b\}\equiv X$, but $\{b\}\cap O\not\equiv X\cap O$ and O is open.

Theorem 6.4. *Let* X *be an* R_0*-space and* $A\equiv B$ *in* X. *If each closed set in* A *is a* G_δ *in* A, *then each closed set in* B *is a* G_δ *in* B.

Proof. Consider $B\cap E$ where E is closed in X. Then $A\cap E$ is closed in A and hence $A\cap E=\cap\{A\cap O_i : i\geqq 1\}$ where each O_i is open in X. It suffices to show that $B\cap E=\cap\{B\cap O_i : i\geqq 1\}$. By Theorem 2.2, $B\cap E\equiv A\cap E$ and since $A\cap E\subseteq O_i$ for each i, it follows that $B\cap E\subseteq O_i$ for each i and thus $B\cap E\subseteq\cap\{B\cap O_i : i\geqq 1\}$. Conversely, let $b\in B\cap O_i$ for each i; it suffices to show that $b\in E$. By Lemma 1.2, $c(b)\cap A\neq\emptyset$; take $a\in c(b)\cap A$. Then $a\in c(b)\subseteq O_i$ and hence $a\in A\cap E$. But $b\in c(a)\subseteq E$ and hence $b\in E$ (in an R_0-space, $a\in c(b)$ implies that $b\in c(a)$).

In Example 1.4, $\{b\}\equiv X$ and $\{b\}$ has the property that each closed set in $\{b\}$ is a G_δ in $\{b\}$. In X, $\{b\}$ is a closed set which is not a G_δ.

Theorem 6.5. *Let* X *be an* R_0*-space and* $A\subseteq X$. *Then* $\cap\{c(a): a\in A\}$ *is the largest set which is equivalent to* A (see Theorem 1.8).

Proof. By Theorem 1.8, it suffices to show that $\cup\{c(a): a\in A\}=A^*$ (see Definition 1.5). Now $a\in A$ implies $c(a)\subseteq O$ when $A\subseteq O$ and O is open. Thus $\cup\{c(a): a\in A\}\subseteq\cap\{O: A\subseteq O,\ O \text{ open}\}=A^*$. Suppose next that $x\notin\cup\{c(a): a\in A\}$. Then $x\in Cc(a)$ for each $a\in A$ and hence $c(x)\subseteq Cc(a)$ since $Cc(a)$ is an open set. It follows then that $c(x)\cap A=\emptyset$

and $A \subseteq Cc(x)$ and $Cc(x)$ is an open set such that $x \notin Cc(x)$. Thus $x \notin A^*$.

In Example 1.4, $\{b\} \equiv X$, and $\bigcup \{c(b) : b \in B\} = B$; but B is not the largest set equivalent to B.

7. R_0-spaces. Connectedness

Theorem 7.1. *Let X be an R_0-space and let $A \equiv B$. If A is connected, then B is connected.*

Proof. Suppose B is disconnected. Then there exist open sets O_1 and O_2 such that $B = (O_1 \cap B) \cup (O_2 \cap B)$, $B \cap O_1 \cap O_2 = \emptyset$ and $B \cap O_1 \neq \emptyset \neq B \cap O_2$. Since $B \subseteq O_1 \cup O_2$, it follows that $A \subseteq O_1 \cup O_2$ and hence $A = (A \cap O_1) \cup (A \cap O_2)$. If $A \cap O_1 = \emptyset$, then $A \subseteq O_2$ which implies that $B \subseteq O_2$. Then $\emptyset \neq B \cap O_1 = B \cap O_2 \cap O_1$ and $B \cap O_2 \cap O_1 \neq \emptyset$, a contradiction. Thus $A \cap O_1 \neq \emptyset \neq A \cap O_2$. Since A is connected, it follows that $A \cap O_1 \cap O_2 \neq \emptyset$; let $a \in A \cap O_1 \cap O_2$. Since X is an R_0-space, $c(a) \cap B \subseteq O_1 \cap O_2 \cap B = \emptyset$ and thus $c(a) \cap B = \emptyset$. Thus $A \not\equiv B$ by Lemma 1.2, a contradiction.

Example 7.2. Let $X = \{a, b, c\}$ with open sets $\emptyset$, $\{a\}$, $\{a, b\}$, $\{a, c\}$, X. Then $\{b, c\} \equiv X$, X is connected and $\{b, c\}$ is disconnected. Thus the R_0 condition cannot be removed from Thcorem 7.1.

Theorem 7.3. *Let X be a space (R_0 not assumed here) and $A \equiv B$ with $A \subseteq B$. If A is connected, so is B.*

Proof. Suppose that $B = (B \cap O_1) \cup (B \cap O_2)$ where O_i is open and $B \cap O_i \neq \emptyset$ and $B \cap O_1 \cap O_2 = \emptyset$. Now $A \subseteq B \subseteq O_1 \cup O_2$ and hence $A = (A \cap O_1) \cup (A \cap O_2)$. If $A \cap O_1 = \emptyset$, then $A \subseteq O_2$ and hence $B \subseteq O_2$; thus $B \cap O_1 \cap O_2 = B \cap O_1 \neq \emptyset$ and $B \cap O_1 \cap O_2 \neq \emptyset$, a contradiction. Thus $A \cap O_1 \neq \emptyset \neq A \cap O_2$. But $A \cap O_1 \cap O_2 \subseteq A \cap CB = \emptyset$ and hence A is disconnected, a contradiction.

Note that in Theorem 7.3, if we assume that B is connected, we cannot deduce that A is connected (see Example 7.2). Note also in Example 7.2 that X is path connected while $\{b, c\}$ is not.

Lemma 7.4. *Let X be an R_0-space and suppose that $f : [0, 1] \to X$ is continuous. Then $g : [0, 1] \to X$ is continuous if $g(t) \in c(f(t))$ for all $t \in [0, 1]$.*

Proof. Let $E \subseteq X$, E closed. It suffices to show that $g^{-1}[E] = f^{-1}[E]$. Now $t \in g^{-1}[E]$ iff $g(t) \in E$ iff $c(g(t)) \subseteq E$ iff $c(f(t)) \subseteq E$ iff $f(t) \in E$

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iff $t \in f^{-1}[E]$. Note that in an R_0-space X, $x \in c(y)$ implies that $y \in c(x)$.

Theorem 7.5. *Let X be an R_0-space and let $A \equiv B$. If A is path connected, then B is path connected.*

Proof. Let $b_1, b_2 \in B$. Take $a_1 \in c(b_1) \cap A$ and $a_2 \in c(b_2) \cap A$. There exists a continuous map $f: [0, 1] \to A$ such that $f(0) = a_1$ and $f(1) = a_2$. Let $g: [0, 1] \to B$ as follows: $g(0) = b_1$ and $g(1) = b_2$, $g(t) \in c(f(t)) \cap B$ for $0 < t < 1$. By Lemma 7.4, g is continuous on B.

Theorem 7.6. *In an R_0-space X, let $A \equiv B$. If A is locally connected, then B is locally connected.*

Proof. Let $b \in O \cap B$, O being open in X. By Lemma 1.2, $c(b) \cap A \neq \emptyset$; let $a \in c(b) \cap A$. Then $a \in c(b) \cap A \subseteqq O \cap A$. Hence there exists a set O^* open in X such that $a \in O^* \cap A \subseteqq O \cap A$ and $O^* \cap A$ is connected. Now $b \in O^* \cap B$ and $O^* \cap A \equiv O^* \cap B$ by Theorem 6.3 and hence $O^* \cap B$ is connected by Theorem 7.1. It suffices then to show that $O^* \cap B \subseteqq O \cap B$. Let $x \in O^* \cap B$; then $c(x) \subseteqq O^*$ and $c(x) \cap A \neq \emptyset$. Let $y \in c(x) \cap A$. Then $y \in O^* \cap A \subseteqq O \cap A$ and hence $x \in c(y) \subseteqq O$. Thus $x \in O \cap B$.

Example 7.7. Let $(X, \mathcal{T})$ be the rationals with the usual topology and $y \notin X$; let $Y = X \cup \{y\}$. Let $\mathcal{U} = \mathcal{T} \cup \{Y\}$. Then $\{y\} \equiv Y$, $\{y\}$ is locally connected, but Y is not. Note that Y is not an R_0-space.

8. R_0, separation

Theorem 8.1. *Let X be an R_0-space and suppose that $A \equiv B$. If A is regular, then B is regular.*

Proof. Let $b \in O \cap B$, O being open in X. Then $c(b) \cap A \neq \emptyset$ by Lemma 1.2; take $a \in c(b) \cap A$. Then $a \in O \cap A$ and hence there exist an open set O^* and a closed set E such that $a \in O^* \cap A \subseteqq E \cap A \subseteqq O \cap A$. It is easy to show that $b \in O^* \cap B \subseteqq E \cap B \subseteqq O \cap B$.

Lemma 8.2. *Let X and Y be spaces, X being an R_0-space. Suppose that $A \equiv B$ in X and that $f: A \to Y$ is continuous. Let $g: B \to Y$ be defined as follows: for $b \in B$, let $g(b) \in f[c(b) \cap A]$. Then $g: B \to Y$ is continuous.*

Proof. Let $E \subseteqq Y$, E closed. Then $f^{-1}[E] = A \cap F$ for some closed set F. It suffices to show that $g^{-1}[E] = B \cap F$ or that $g^{-1}[E] \subseteqq F$. Let $b \in g^{-1}[E]$. Then $g(b) \in E$ and $g(b) \in f[c(b) \cap A]$ or $g(b) = f(a)$ where $a \in c(b) \cap A$. Then $f(a) \in E$ and hence $a \in f^{-1}[E] \subseteqq F$. $b \in c(a) \subseteqq F$.

Thcorem 8.3. *Let $A \equiv B$ in an R_0-space X. If A is completely regular, then B is completely regular.*

Proof. Let $b \in O \cap B$, O being open in X. By Lemma 1.2, $c(b) \cap A \neq \emptyset$; take $a \in c(b) \cap A$. It follows that $a \in O \cap A$. Since A is completely regular, there exists a continuous map $f : A \to [0, 1]$ such that $f(a)=0$ and $f(a^*)=1$ for all $a^* \in A-O$. Let $g : B \to [0, 1]$ be as in Lemma 8.2, $g(b)$ being taken as $f(a)$. Then $g(b)=f(a)=0$. Now let $b^* \in B-O$. Then $c(b^*) \subseteq CO$ and $a^* \in c(b^*) \cap A$ which implies that $a^* \in A-O$ and thus $g(b^*)=f(a^*)=1$. $g : B \to [0, 1]$ is continuous by Lemma 8.2.

In Example 1.4. $\{b\} \equiv X$, $\{b\}$ is completely regular, but X is not completely regular.

Theorem 8.4. *Let $A \equiv B$ in an R_0-space X. If A is normal, then B is normal.*

Proof. Let $B \cap F \cap E = \emptyset$, E and F being closed in X. Then $B \subseteq \mathcal{C}(E \cap F)$, an open set, and hence $A \subseteq \mathcal{C}(E \cap F)$ and $A \cap E \cap F = \emptyset$. Since A is normal, there exist open sets O_1 and O_2 in X such that $A \cap E \subseteq A \cap O_1$ and $A \cap F \subseteq A \cap O_2$ and $A \cap O_1 \cap O_2 = \emptyset$. Applying Theorem 2.2, it follows that $B \cap E \subseteq B \cap O_1$ and $B \cap F \subseteq B \cap O_2$. It remains to show that $B \cap O_1 \cap O_2 = \emptyset$. Suppose $b \in B \cap O_1 \cap O_2$. Take $a \in c(b) \cap A$; then $a \in O_1 \cap O_2 \cap A$, a contradiction.

In Example 7.2, $\{b, c\} \equiv X$, $\{b, c\}$ is normal, but X is not.

Theorem 8.5. *Let A be a completely normal subspace of an R_0-space X. Then $\cup \{c(a) : a \in A\}$ is completely normal.*

Proof. Let $B \subseteq \cup \{c(a) : a \in A\}$. We must show that B is normal. Let $A^\# = \{a : c(a) \cap B \neq \emptyset,\ a \in A\}$. It suffices to show that $\cup \{c(b) : b \in B\} = \cup \{c(a) : a \in A^\#\}$ since $A^\#$ is normal and is equivalent to $\cup \{c(a) : a \in A^\#\}$ and B is equivalent to $\cup \{c(b) : b \in B\}$ (see Theorem 8.4). Let $x \in c(b)$ for some $b \in B$. Then $b \in c(a)$ for some $a \in A$. Then $x \in c(b) \subseteq c(a)$ and hence $a \in A^\#$. Thus $x \in \cup \{c(a) : a \in A^\#\}$. Conversely, let $y \in c(a)$ for some $a \in A^\#$. Then $c(a) \cap B \neq \emptyset$; let $b \in c(a) \cap B$. They $y \in c(a) \subseteq c(b)$.

Corollary 8.6. *In an R_0-space X, let $A \equiv B$, A being completely normal. Then B is completely normal.*

Proof. By Theorem 8.5, $\cup \{c(a) : a \in A\}$ is completely normal and by Theorem 6.5, $B \subseteq \cup \{c(a) : a \in A\}$. Hence B is completely normal.

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9. R_0 and conutability

Theorem 9.1. *Let $A \equiv B$ in an R_0-space X. Ij A is separable, then B is separable.*

Proof. Let $\{a_n : n \geqq 1\}$ be dense in A. Take $b_n \in c(a_n) \cap B$ for each $n \geqq 1$. Then $\{b_n : n \geqq 1\}$ is dense in B; let $\emptyset \neq O \cap B$ were O is open in X. Choose $b \in O \cap B$ and let $a \in c(b) \cap A$. Then $a \in O \cap A$. Since $O \cap A \neq \emptyset$, $a_n \in O \cap A$ for some n. It follows then that $b_n \in O \cap B$.

Example 9.2. Let $(X, \mathcal{T})$ be an uncountable discrete space and $y \notin X$; let $Y = X \cup \{y\}$ and $\mathcal{U} = \mathcal{T} \cup \{Y\}$. Then $\{y\} \equiv Y$, $\{y\}$ is separable, but Y is not separable.

Theorem 9.3. *In an R_0-space X, let $A \equiv B$ and let A be a second axiom space. Then B is second axiom.*

Proof. If $\{A \cap O_i : i \geqq 1,\ O_i \text{ open in } X\}$ is a base for $A \cap \mathcal{T}$, then $\{B \cap O_i : i \geqq 1\}$ is a base for $B \cap \mathcal{T}$.

In Example 9.2, $\{y\}$ is second axiom, but Y is not.

10. R_0 and local compactness, paracompactness

Lemma 10.1. *In an R_0-space X, let A be locally compact. Then $\cup \{c(a) : a \in A\}$ is locally compact.*

Proof. Let $x \in \cup \{c(a) : a \in A\}$. Then $x \in c(a^*)$ for some $a^* \in A$ and hence there exists an open set O and a compact set M such that $a^* \in O \cap A \subseteqq M \subseteqq A$. Then $x \in O \cap \cup \{c(a) : a \in A\} \subseteqq \cup \{c(m) : m \in M\} \subseteqq \cup \{c(a) : a \in A\}$. Now $M \equiv \cup \{c(m) : m \in M\}$ and since M is compact, so is $\cup \{c(m) : m \in M\}$ (see Theorems 6.5 and 4.1).

Lemma 10.2. *In an R_0-space X, A is locally compact if $\cup \{c(a) : a \in A\}$ is locally compact.*

Proof. Let $a^* \in A$; there exists an open set O and a compact set M such that $a^* \in O \cap \cup \{c(a) : a \in A\} \subseteqq M \subseteqq \cup \{c(a) : a \in A\}$. Then $a^* \in O \cap A \subseteqq A \cap \cup \{c(m) : m \in M\} \subseteqq A$. We need only show that $A \cap \cup \{c(m) : m \in M\}$ is compact. By Theorem 4.1 and Theorem 6.5, it suffices to show that $\cup \{c(a') : a' \in A \cap \cup \{c(m) : m \in M\}\}$ is compact. The reader can easily verify that this set is merely $\cup \{c(m) : m \in M\}$ which is compact.

Theorem 10.3. *In an R_0-space, let $A \equiv B$. If A is locally compact, then B is locally compact* (see Theorem 4.3).

Proof. A locally compact implies that $\cup\{c(a): a\in A\}$ is locally compact (Lemma 10.1). But $\cup\{c(a): a\in A\}=\cup\{c(b): b\in B\}$. Hence B is locally compact by Lemma 10.2.

Theorem 10.4. *In an R_0-space X, let $A\equiv B$ and suppose that A is paracompact. Then B is paracompact.*

Proof. Suppose that $B=B\cap\cup\{O_\alpha:\alpha\in\triangle\}$ where each O_α is open in X. Then $B\subseteqq\cup\{O_\alpha:\alpha\in\triangle\}$ and hence $A\subseteqq\cup\{O_\alpha:\alpha\in\triangle\}$. There exists then a family of open sets $\{O_\gamma:\gamma\in\Gamma\}$ such that $A=\cup\{A\cap O_\gamma:\gamma\in\Gamma\}$, $\{A\cap O_\gamma:\gamma\in\Gamma\}$ is locally finite in A and $\{A\cap O_\gamma:\gamma\in\Gamma\}$ refines $\{A\cap O_\alpha:\alpha\in\triangle\}$. Thus $B=\cup\{B\cap O_\gamma:\gamma\in\Gamma\}$, $\{B\cap O_\gamma:\gamma\in\Gamma\}$ is locally finite in B and $\{B\cap O_\gamma:\gamma\in\Gamma\}$ refines $\{B\cap O_\alpha:\alpha\in\triangle\}$. The details are left to the reader.

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