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PLASTERABLE CONES IN LOCALLY CONVEX HAUSDORFF SPACES

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1. Let (E, τ, K) be a locally convex Hausdorff space ordered by a cone K. (Recall that a set K is a cone if and only if $K+K\subseteq K$, $\lambda K\subseteq K$ $(\lambda\geq 0)$, and $K\cap -K=\{0\}$.) A linear functional f defined on (E, τ, K) is called positive if $f(x)\geq 0$ $(x\in K)$. A positive linear functional f is called *strictly positive* if $f(x)>0$ $(0\neq x\in K)$. We say f is *uniformly positive* on K if for every continuous seminorm p there exists a positive number a_p such that $f(x)\geq a_p p(x)$ $(x\in K)$. A cone K *allows plastering* by a cone K_1 if there exists a family P of seminorms generating τ such that each $0\neq x\in K$ is an interior point of K_1, and furthermore, for each $p\in P$ there exists $a_p>0$ such that $0\neq x_0\in K$ implies $\{x_0+h: p(h)\leq a_p p(x_0)\}\subseteq K_1$. We sometimes say K *is plasterable by* K_1. A cone K has a *base* if and only if there exists a nonempty convex set B such that each $0\neq x\in K$ has a unique representation of the form $x=\lambda y$ $(\lambda>0,\ y\in B)$. It is known, see [5, Prop. 3.6, p. 26], that a subset B of a vector space E ordered by a cone K is a base for K if and only if there is a strictly positive linear functional f defined on E such that $B=f^{-1}(1)\cap K$. This motivates the following defintion, see [4]. A subset B of an ordered locally convex space (E, τ, K) is a *hyperbase* for K if and only if there is a strictly positive continuous linear functional defined on E such that $B=f^{-1}(1)\cap K$. The definitions of a uniformly positive linear functional and a cone being plasterable are abstractions of Banach space definitions stated by Krasnoselskii [2, pp. 31—32]. In [2, p. 32] Krasnoselskii shows for a closed cone K in a Banach space that the existence of a uniformly positive continuous linear functional and K plasterable are equivalent. In this paper the above mentioned theorem of Krasnoselskii is extended to ordered locally convex Hausdorff spaces and relationships between hyperbasis, positive continuous linear functionals, and generating families of seminorms with certain properties are examined.

2. We begin with the following result.

Theorem 1. *Let (E, τ, K) be an ordered locally convex Hausdorff space with topological dual E'. The following are equivalent.*

a) *K has a hyperbase.*

b) *There exists $f\in E'$ such that f is strictly positive on K.*

c) *There exists a cone K_1 such that each nonzero element of K is an interior point of K_1.*

If, in addition, (E, τ, K) is separable and barrelled, the above are equivalent to the following.

d) *τ is generated by a family $P=\{p\}$ of seminorms with the property that for each $0\neq x_0\in K$ and $p\in P$ there exists $\varepsilon>0$ for which $\{x : p(x-x_0)<\varepsilon\}\cap -K=\emptyset$.*

Proof. The equivalence of a) and b) is simply the definition of K has a hyperbase. To show b) implies c), suppose $f\in E'$ and f is strictly positive on K. It follows easily that $K_1=\{x\in E\,|\,x=ty\ (t>0, y\in f^{-1}(1))\}$ is a cone such that each $0\neq x\in K$ is an interior point of K_1. If there exists a cone K_1 such that every nonzero element of K is contained in the interior of K_1, then the existence of a nonzero $f\in E'$ such that $f(x)>0$ $(x\in K)$ is guaranteed by [6, p. 29]. Thus c) implies b).

Now assume τ is generated by a family $P=\{p\}$ of seminorms with the property described in d). Choose some $p\in P$ and let $S=\{x : p(x)\leq 1\}$. Since $S^0=\{f\in E'\,|\,|f(x)|\leq 1\ (x\in S)\}$ is $\sigma(E', E)$-compact and $K'=\{f\in E'\,|\,f(x)\geq 0\,(x\in E)\}$ is $\sigma(E', E)$-closed, $S^0\cap K'$ is a $\sigma(E', E)$ compact subset of E'. Let $\{x_n\,|\,n\in N\}$ be a countable dense subset of E, and define a metric on $S^0\cap K'$ by

$$d(f_1, f_2)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{|f_1(x_n)-f_2(x_n)|}{1+|f_1(x_n)-f_2(x_n)|}\,.$$

The metric defines a Hausdorff topology on $S^0\cap K'$ which is weaker than the $\sigma(E', E)$-topology on $S^0\cap K'$. Thus the metric topology is equivalent to the $\sigma(E', E)$-topology on $S^0\cap K'$ by [1, p. 18]. Therefore $S^0\cap K'$ with the $\sigma(E', E)$-topology is a compact metric space. Let $\{f_n\,|\,(n\in N)\}$ be a countable dense subset of $S^0\cap K'$. For each $x\in E$, define f_0 by $f_0(x)=\sum_{n=1}^{\infty}\frac{1}{2^n}f_n(x)$. Since E' is $\sigma(E', E)$-sequentially complete, $f_0\in E'$. If $0\neq x\in E$, then the condition on p given in d) implies $x\notin\sigma(E, E')$ closure of $-K$. Hence there exists an n_0 such that $f_{n_0}(x)>0$. Thus f is a strictly positive continuous linear functional and $\{x : f_0(x)=1\}\cap K$ is a hyperbase for K. Thus d) implies b). On the other hand, if $f\in E'$ is strictly positive on K, then $\{p\,|\,p$ is a continuous seminorm and $p(x)\geq|f(x)|\,(x\in E)\}$ is a generating family of seminorms for τ which satisfies the condition in d). Thus b) implies d).

The following lemma is stated without proof in [2, p. 30] for real Banach spaces. It actually holds in any real normed linear space. The proof is straightforward and is omitted.

Lemma 2. *Let* $(E, \|\cdot\|)$ *be a real normed linear space,* $f\in E'$, *and* $x_0\in E$. *Then* $|f(x_0)|=\inf\{\|x_0-y\|: y\in f^{-1}(0)\}$.

The above lemma is used to prove the following theorem.

Theorem 3. *Let* (E, τ, K) *be an ordered locally convex Hausdorff space. The following are equivalent.*

a) K *has a bounded hyperbase.*

b) *There exists* $f\in E'$ *such that* f *is uniformly positive on* K.

c) K *allows plastering by a cone* K_1.

Proof. To prove a) $\Rightarrow$ b), suppose p is a continuous seminorm and K has a bounded hyperbase $B=f^{-1}(1)\cap K$, where $f\in E'$ is strictly positive. If $y\in K$ and $y\neq 0$, then there exists $x\in B$ and $\lambda>0$ such that $y=\lambda x$. If $p(z)=0$ $(z\in B)$, then $f(y)\geq a_p\lambda p(x)=0$ for any $a_p>0$. If $p(z)\neq 0$ for some $z\in B$, then $f(y)=f(\lambda x)=\lambda f(x)=\lambda\geq\dfrac{1}{\sup\{p(z)\,|\,z\in B\}}\,p(y)$. Thus f is uniformly positive on K.

Suppose $f\in E'$, f is uniformly positive on K, and Q is the family of all continuous seminorms on (E, τ). Let $N=\{x\in E\,|\,f(x)=1\}$, then $K_1=\{y\in E\,|\,y=tx\ (t\geq 0,\ x\in N)\}$ is a plastering for K. It is clear that K is a cone and $K\subseteq K_1$. For each $q\in Q$ let a_q be such that $f(x)\geq a_q q(x)$ $(x\in K)$. It then follows that $P=\{p\in Q: p(x)\geq|f(x)|\,(x\in E)\}$ is a τ-generating family of seminorms, and $N_p(x_0)=\{x+h: h\in E$ and $p(h)\leq \dfrac{a_p|f(x_0)|}{2}\}\subseteq K_1$ for each $0\neq x_0\in K$. For $0<f(x_0)<p(x_0)$ implies $f(x_0+h)\geq f(x_0)-|f(x)|\geq a_p p(x_0)-p(h)\geq a_p p(x_0)-a_p\dfrac{p(x_0)}{2}=\dfrac{a_p p(x_0)}{2}>0$, and therefore $\dfrac{x_0+h}{f(x_0+h)}\in N$. Hence b) $\Rightarrow$ c).

To prove that c) $\Rightarrow$ a) suppose K is plasterable by a cone K_1. Let $Q=\{q\}$ be a generating family of seminorms such that for each q there exists $a_q>0$ for which $\{x_0+h\,|\,q(h)\geq a_q q(x_0)\}\subseteq K_1 (0\neq x_0\in K)$. By [6, p. 64] there exists a nonzero $f\in K'$ such that the interior of K_1 is contained in $\{x\in E: f(x)>0\}$. Therefore $K_1\subseteq\{x\in E: f(x)\geq 0\}=W$, and f is strictly positive on K. Therefore $f^{-1}(1)\cap K=B$ is a base for K. It remains to show that B is bounded. For each $q\in Q$ define p by $p(x)=\sup\{|f(x)|, q(x)\}$. Let $P=\{p\}$ denote the family of seminorms obtained by letting q range over Q, and let a_p denote $\min\{a_q, 1\}$. P generates τ. Furthermore, $\{x_0+h : p(h)\leq a_p p(x_0)\}\subseteq W$ for each $0\neq x_0\in K$, since $0<f(x_0)<p(x_0)$ and $p(h)\leq a_p p(x_0)$ imply $f(x_0+h)>\frac{a_p p(x_0)}{2}>0$. Therefore $x_0+h\in W$, since $\frac{x_0+h}{f(x_0+h)}\in N$ and $f(x_0+h)>0$.

Suppose that K is plasterable by a cone K_1. Let $Q=\{q\}$ be a generating family of seminorms such that for each q there exists $a_q>0$ for which $\{x_0+h\,|\,q(h)\leq a_q q(x_0)\}\subseteq K_1$ $(0\neq x_0\in K)$. By [6, p. 64] there exists a nonzero $f\in E'$ such that the interior of K_1 is contained in $\{x\in E\,|\,f(x)>0\}$. Since K is contained in the interior of K_1, f is strictly positive on K. Therefore $f^{-1}(1)\cap K=B$ is a hyperbase for K. It remains to show that B is bounded, and hence that (c) implies (a). For each $q\in Q$ define p by $p(x)=\sup\{|f(x)|, q(x)\}$ and a_p by $a_p=\min\{a_q, 1\}$. Let $P=\{p\}$ be the family of seminorms obtained in the above manner by letting q range over all seminorms in Q. P generates τ, and $\{x_0+h : p(h)\leq a_p p(x_0)\}\subseteq W$ for each $0\neq x_0\in K$. Let $E_p=\{[x]\,|\,[x]=\{x_0+h\,|\,p(h)=0\}\}$ topologized by the norm $\|[x]\|_p=p(y)$ $(y\in[x])$. Since for each $p\in P$ $p(x)\geq|f(x)|$ $(x\in E)$, the functional f_p defined on E_p by $f_p([x])=f(y)$ $(y\in[x])$ is linear and $f_p\in E_p'$ $(p\in P)$. Let $\|f_p\|_p$ denote the norm of the linear functional f_p with domain E_p. We now show, for each $p\in P$, that $f(x)\geq\frac{a_p}{2}p(x)$ $(x\in K)$. Suppose $x_0\in K$ and $p(x_0)\neq 0$. Then by Lemma 2, $f(x_0)=f_p([x_0])=\|f_p\|_p \inf\{\|[x_0]-[y]\|_p\,|\,[y]\in f_p^{-1}(0)\}=\|f_p\|_p \inf\{p(x_0-y)\,|\,[y]\in f_p^{-1}(0)\}=\|f_p\|_p \inf\{p(x_0-y)\,|\,y\in f^{-1}(0)\}$. If $\inf\{p(x_0-y)\,|\,y\in f^{-1}(0)\}<a_p p(x_0)$, then $p(x_0-x_1)<a_p p(x_0)$ for some $x_1\in f^{-1}(0)$. However, $p(x_0-x_1)<a_p p(x_0)$ implies $\{x\in E\,|\,p(x-x_1)\leq a_p p(x_0)-p(x_0-x_1)\}\subseteq W$, since $p(x-x_1)\leq a_p p(x_0)-p(x_0-x_1)$ implies $p(x_0-x)\leq p(x_0-x_1)+a_p p(x_0)-p(x_0-x_1)=a_p p(x_0)$. We therefore have x_1 belongs to the interior of W, and so $f(x_1)>0$. This is a contradiction as $x_1\in f^{-1}(0)$. Therefore $\inf\{p(x_0-y)\,|\,y\in f^{-1}(0)\}\geq a_p p(x_0)$. It therefore follows that $f(x_0)\geq\|f_p\|_p a_p p(x_0)$ $(0\neq x_0\in K)$. Hence $f^{-1}(1)\cap K=B$ is bounded, since $\frac{1}{\|f_p\|_p a_p}\geq p(x)$ $(x\in B)$.

It shall be pointed out that Theorem 3 is weaker than Krasnoselskii's in the following sense. Although Theorem 3 does hold in Banach spaces with the additional requirement that a cone be closed, we are not assuming in this paper that a cone is necessarily closed. Furthermore, it is unknown to the author if in a locally convex space a closed cone with a bounded hyperbase necessarily allows plastering by a closed cone. In Krasnoselski's result the cone K_1 which plasters the cone K also has a bounded hyperbase and hence K_1 is normal. But a locally convex space ordered by a normal cone with nonempty interior is necessarily normable [5, p. 67]. Hence Krasnoselski's proof will not generalize to obtain a closed cone K_1 which plasters K. We formalize the above discussion in the following theorem.

Theorem 4. *Let (E, τ) be a locally convex space ordered by a closed cone K, which allows plastering then K allows plastering by a closed cone with a bounded hyperbase if and only if (E, τ) is normable.*

Furthermore, it is unknown to the author if in a locally convex space a closed cone with a bounded hyperbase necessarily allows plastering by a closed cone K_1.

3. We conclude with some examples of cones in locally convex spaces which allow plastering.

Example 1. Let (s) be the space of all real sequences topologized in the usual manner, and $K=\{x\in(s)\ x_1\geq x_i\geq 0\ (i=1, 2, \cdots\cdots)\}$. The linear functional f defined by $f(x)=x_1$ $(x=(x_i))$ is strictly positive on K and $\{x\in K\,|\,x_1=1\}$ is a bounded hyperbase for K.

Example 2. Let (E, τ) be a locally convex Hausdorff space and furthermore suppose that x_1 is a nonzero element of E. Then there exists $f\in E'$ such that $f(x_1)=1$. Define the projection T from E onto the subspace spanned by x_1 by $T(x)=f(x)x_1$ $(x\in E)$. Let P be a generating family of continuous seminorms on (E, τ) such that for each $p\in P$, $p(x)\geq |f(x)|$ $(x\in E)$. Let $K=\{x\in E: f(x)\geq 0$ and $q(Tx)\geq q((I-T)x)$ $(q\in P)\}$. K is a closed cone with a bounded hyperbase.

Example 3. As a special case of Example 2 one might consider a locally convex Hausdorff space (E, τ) with Schauder basis (x_n, f_n). Let $P=\{p\}$ be a generating family of τ-continuous seminorms such that $p(x)\geq |f_1(x)|$ $(x\in E)$. Define $K=\{\sum_{n=1}^{\infty} f_n(x)x_n\in E\,|\,f_1(x)\geq 0$ and $p(f_1(x)x_1)\geq p(\sum_{n=2}^{\infty} f_n(x)x_n)$ $(p\in P)$.

Example 4. A yet more restrictive case of the above is the Lorentzian cone [3, pp. 48—53] in ℓ_2. In this case $K=\{(x_i)=x\in\ell_2 : x_1\geq 0$ and $x_1\geq\sqrt{\sum_{i=2}^{\infty}x_i^2}\}$.

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