Mathematical Journal of Okayama University

Volume 12, *Issue* 2 1963 *Article* 8

OCTOBER 1966

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Takasi Nagahara[*]

[*] Okayama University

SUPPLEMENTS TO THE PREVIOUS PAPER "QUASI-GALOIS EXTENSIONS OF SIMPLE RINGS"

TAKASI NAGAHARA

This paper is about quasi-Galois conditions (abr. q-Galois conditions) of simple rings. In [3], we have presented some q-Galois conditions which are useful in the study of q-Galois extensions. In this paper, we shall present more explicit one of q-Galois conditions. Moreover, as appendices, we shall make a remark on two-sided linear homomorphism rings of q-Galois extensions which is connected with Elliger's quasi-Galois extensions [1].

Throughout the present paper, $R=\sum De_{ij}$ will be a simple ring, where e_{ij}'s are matrix units and D is the centralizer of $\{e_{ij}\}$ in R which is a division ring. Moreover, we shall be concerned with a fixed subring S of R which is a simple ring containing the identity element 1 of R. For these rings R, S, we set $V=V_R(S)$, $H=V_R^2(S)$, and let $\mathfrak{R}$, $\mathfrak{R}_{l.f}$, etc. be as in [3]. In addition, we denote by C, Z, C_0 the centers of R, S, V respectively. As to notations and terminologies used in this note, we follow the previous paper [3].

In the discussion of q-Galois conditions as in [2], it may well be presented as fundamental theme that $\mathrm{Hom}_{S_l}(S', R)=\mathfrak{G}(S'/S, R)R_r$ for every $S'\in\mathfrak{R}_{l.f}$) (i.e., R is q-Galois over S if in addition $S\in\mathfrak{R}$) if and only if $J(\mathfrak{G}(S'/S, R), S')=S$ for every $S'\in\mathfrak{R}_{l.f}$. This equivalence will be concerned in the main theme of our study (Principal Theorem). In [4], H. Tominaga has proved this equivalence for the case that R is a division ring and R is left locally finite over S ([4, Remark 2]).

First, we shall prove the following lemma whose proof is partially similar to that of [4, Remark 2].

Lemma 1. *Let R be left locally finite over S, and $S\subset C$. If $J(\mathfrak{G}(T/S, R), T)=S$ for every $T\in\mathfrak{R}^0_{l\cdot f}$ then R is q-Galois over S.*

Proof. By [3, Theorem 3], it will suffice to prove that $\mathrm{Hom}_{S_l}(T, R)=\mathfrak{G}(T/S, R)R_r$ for every $T\in\mathfrak{R}^0_{l.f}$. Let $T\in\mathfrak{R}^0_{l.f}$. First, we shall show that for elements a, b of T such that $\{a, b\}$ is linearly independent over S and a is regular, there exists a $\rho\in\mathfrak{G}(T/S, R)R_r$ such that

$$a\rho=0, \quad b\rho=b.$$

Since $ba^{-1}\in T\setminus S$ and $J(\mathfrak{G}(T/S, R), T)=S$, there exists some $\sigma\in\mathfrak{G}(T/S, R)$

with $(ba^{-1})\sigma\neq ba^{-1}$. Then, for $\tau=1-\sigma(a\sigma)^{-1}{}_r a_r$ we see that $a\tau=0$, $b\tau\neq 0$. Since R is $\langle R\rangle R_r$-irreducible, it follows that $(b\tau)\langle R\rangle R_r=R$, and so $b\rho=b$ for some $\rho\in\tau\langle R\rangle R_r\subset\mathfrak{G}(T/S, R)R_r$ (note $R=V$). Moreover we have $a\rho\in(a\tau)\langle R\rangle R_r=\{0\}$. As is well-known, T is generated by the set of regular elemsnts of T. Hence, to prove our assertion, it will suffice to show that for regular elements $x_1, \cdots, x_n$ of T such that $\{x_1, \cdots, x_n\}$ is linearly independent over S, there exists a $\rho\in\mathfrak{G}(T/S, R)R_r$ such that

$$x_i\rho=0,\ i=1, \cdots, n-1;\ x_n\rho=x_n.$$

For $n=2$, our assertion holds by the above remark. Assume that we can find $\rho_1, \cdots, \rho_{n-1}\in\mathfrak{G}(T/S, R)R_r$ such that $x_i\rho_j=\delta_{ij}x_i$, $i=1, \cdots, n-1$, where $\delta_{ii}=1$, $\delta_{ij}=0$ for $i\neq j$. There holds then $x_i(\sum\rho_j-1)=0$ for $i=1, \cdots, n-1$. If $x_n(\sum\rho_j-1)\neq 0$ then $(x_n(\sum\rho_j-1))\langle R\rangle R_r=R$, and so $x_n\rho=x_n$ for some $\rho\in(\sum\rho_j-1)\langle R\rangle R_r\subset\mathfrak{G}(T/S, R)R_r$. Moreover we have $x_i\rho=0$, $i=1, \cdots, n-1$. If otherwise $x_n(\sum\rho_j-1)=0$ then $\sum_j x_n\rho_j=x_n$. Since $\{x_1, \cdots, x_n\}$ is linearly independent over S, some $\{x_i, x_n\rho_i\}$ is linearly independent over S. As $\rho_i\in\mathfrak{G}(T/S, R)R_r$, we write $\rho_i=\sum_{k=1}^m\sigma_k a_{kr}$, where $\sigma_k\in\mathfrak{G}(T/S, R)$, $a_k\in R$, $k=1, \cdots, m<\infty$. Then, for $T'\in\mathfrak{R}^0_{l.f}/\{T, T\rho_i, T\sigma_1, \cdots, T\sigma_m, a_1, \cdots, a_m\}$ there exists a $\rho'\in\mathfrak{G}(T'/S, R)R_r$ such that $x_i\rho'=0$, $x_n\rho_i\rho'=x_n\rho_i$ (by the first remark in the proof). From $(x_n\rho_i\rho')\langle R\rangle R_r=R$ we have $x_n\rho_i\rho''=x_n$ for some $\rho''\in\rho'\langle R\rangle R_r\subset\mathfrak{G}(T'/S, R)R_r$. Clearly $x_i\rho''=0$. If we set $\rho=\rho_i\rho''$ then $\rho\in\mathfrak{G}(T/S, R)R_r$, $x_n\rho=x_n$, $x_j\rho=x_j\rho_i\rho''=0$ for $j\neq i$, and $x_i\rho=x_i\rho_i\rho''=x_i\rho''=0$. This completes the proof.

Now, as in [3, Lemma 11], for each $T\in\mathfrak{R}^0_{l.f}$, we set $\tilde{\mathfrak{G}}(T/S, R)=\bigcap_{T'}\mathfrak{G}(T'/S, R)\,|\,T$, where T' runs over all the elements of $\mathfrak{R}^0_{l.f}/T$. Then, we have the following

Corollary 1. *Let R be left locally finite over S, and $S\in\mathfrak{R}$. If $J(\mathfrak{G}(T/S, R), T)=S$ for every $T\in\mathfrak{R}^0_{l.f}$ then V is q-Galois and locally finite over Z, and $(C_0\cap T)\tilde{\mathfrak{G}}(T/S, R)\subset C_0$ for every $T\in\mathfrak{R}^0_{l.f}$.*

Proof. Since $[S, V]=S\times_Z V$ (direct product) and $(T\cap V)\mathfrak{G}(T/S, R)\subset V$ for every $T\in\mathfrak{R}^0_{l.f}$, it follows that V is locally finite over Z, and $J(\mathfrak{G}(V^*/Z, V), V^*)=Z$ for every regular subring V^* of V/Z with $[V^*:Z]<\infty$ and $[V^*|V^*]=[V|V]$. Hence, by Lemma 1, V is q-Galois over Z. Now, let $T\in\mathfrak{R}^0_{l.f}$, and let $\{f_{ij}\}$ be a system of matrix units of V such that $V_V(\{f_{ij}\})$ is a division ring. Then, for $T'\in\mathfrak{R}^0_{l.f}/T\cup\{f_{ij}\}$, $V\cap T'$ is a regular subring of V/Z with $[V\cap T':Z]<\infty$ and $[V\cap T'|V\cap T']=[V|V]$. Hence, we see that $\mathfrak{G}(T'/S, R)\,|\,V\cap T'\subset\mathfrak{G}(V\cap T'/Z, V)$. From this and $\tilde{\mathfrak{G}}(T'/S, R)\,|\,T=\tilde{\mathfrak{G}}(T/S, R)$ ([3, Lemma 12]), it follows that $(C_0\cap T)\tilde{\mathfrak{G}}(T/S, R)=(C_0\cap T)(\tilde{\mathfrak{G}}(T'/S, R)\,|\,T)=(C_0\cap T)(\tilde{\mathfrak{G}}(T'/S, R)\,|\,V\cap T')\subset$

$(C_0 \frown T)\mathfrak{G}(V \frown T'/Z, V)$. Since V is q-Galois over Z, we have $(C_0 \frown T)\cdot \mathfrak{G}(V \frown T'/Z, V) \subset C_0$ by [3, Theorem 3]. This implies our last assertion.

Remark 1. If R is left locally finite over S, and $S \in \mathfrak{R}$ then $[S, V]$ is a simple ring. Indeed, for every intermediate simple ring V_ι of V/Z with $[V_\iota : Z] < \infty$, $[S, V_\iota]$ is a simple ring with minimum condition. Since $[[S, V_\iota] \,|\, [S, V_\iota]] \leqq [R \,|\, R]$, $[S, V]$ is also a simple ring with minimum condition (cf. [3, Remark 11]). Moreover, for an arbitrary non-zero element a of $[S, C_0]$, $SaS \frown C_0$ contains non-zero (regular) elements. Hence, every intermediate ring of $[S, C_0]/S$ is simple. In particular, $[S, C_0]$ is a simple ring.

Lemma 2. *Let R be left locally finite over S, and $S \in \mathfrak{R}$. Then R is left locally finite over $[S, C_0]$, and $[H \frown [T, V] : [S, C_0]]_l < \infty$ for every $T \in \mathfrak{R}^0_{l.f}$.*

Proof. For $T \in \mathfrak{R}^0_{l.f}$, we set $V' = V_R(T)$. Then, by [3, Lemma 1(iv)], we have $[V : V']_l \leqq [T : S]_l < \infty$, and we set $V = \sum_{i=1}^{p} V'd_i$ $(p < \infty)$. Moreover, we set $[T, d_1, \cdots, d_p] = \sum_{j=1}^{q} Tf_j$ $(q < \infty)$. Then, for all integer s, we have that $(V \cdot T)^s \subset V' \cdot [T, d_1, \cdots, d_p] = V' \cdot (\sum_{j=1}^{q} Tf_j) \subset \sum_{j=1}^{q} [V', T] f_j$. Hence $[V, T] = \sum_{j=1}^{q} [V', T] f_j$, and so $[[V, T] : [V', T]]_l < \infty$. Let C^* be the center of V', and set $H^* = H \frown [V, T] (= V_{[V,T]}(V))$. Then $[V', T] = V' \times_{V' \frown T} T$ (direct product) $= V' \times_{C^*} [C^*, T] \subset V' \times_{C^*} [C^*, T, H^*] \subset [V, T]$. Hence we see that

(1) $$[[C^*, T, H^*] : [C^*, T]]_l < \infty.$$

Now, we set $C_0^* = C_0 \frown C^*$. Then one will easily see that C_0^* is the center of $[V, T]$. Hence, by [3, Lemma 1 (iv)], we have $[V_{[V,T]}([V', T]) : C_0^*] \leqq [[V, T] : [V', T]]_l < \infty$. Since $V_{[V,T]}([V', T]) = C^*$, we obtain $[C^* : C_0^*] < \infty$. Therefore, it follows that

(2) $$[[C^*, T] : [C_0^*, T]]_l < \infty.$$

Further, $[C_0^*, S]$ is a simple ring by Remark 1, and we have

(3) $$[[C_0^*, T] : [C_0^*, S]]_l < \infty.$$

Combining (1), (2), and (3), it follows that

$$[[C^*, T, H^*] : [C_0^*, S]]_l < \infty.$$

Noting that $H^* \supset [C_0, S] \supset [C_0^*, S]$, we obtain the following

$$[H^* : [C_0, S]]_l < \infty,$$
$$[[C_0, T] : [C_0, S]]_l < \infty.$$

The last form implies that R is left locally finite over $[C_0, S]$.

Lemma 3. *Let R be left locally finite over S. Let $T, T_1 \in \mathfrak{R}^0_{l.f}$ and*

$\sigma_1 \in \tilde{\mathfrak{S}}(T_1/S, R)$. Then there exists some $T_2 \in \mathfrak{R}^0_{l.f}/T \cup T_1$ and some $\sigma_2 \in \tilde{\mathfrak{S}}(T_2/S, R)$ such that $\sigma_2 | T_1 = \sigma_1$, and if $\tau \in \mathfrak{S}(T/S, R)$ then $\tau' \sigma_2 \sim \tau$ for some $\tau' \in \mathfrak{S}(T/S, R)$, and in particular $\tau'' \sigma_2 \sim 1$ for some $\tau'' \in \mathfrak{S}(T/S, R)$[1].

Proof. By [3, Corollary 2 (i)], we have $(\mathfrak{S}(T/S, R) : \langle V \rangle) < \infty$. Hence we can write $\mathfrak{S}(T/S, R) = \bigcup_{i=1}^{m} \tau_i \langle V \rangle$, $m < \infty$, and $\mathfrak{S}(T/S, R) R_r = \sum_{i,j} \tau_i \langle v_{ij} \rangle R_r$. Let $R = \sum_{i,j=1}^{n} e_{ij} D$ as in the introduction. Then for $f \in \mathrm{Hom}_{S_l}(T, R)$, $f e_{ii} R_r$ is 0 or R_r-irreducible, i. e., $[f e_{ii} R_r | R_r] \leqq 1$, $i = 1, \cdots, n$. Since $[\mathrm{Hom}_{S_l}(T, R) | R_r] = [T : S]_l [R | R] < \infty$, it follows that

$$\begin{aligned} \mathrm{Hom}_{S_l}(T, R) &= \mathfrak{S}(T/S, R) R_r \oplus \textstyle\sum_{\oplus k=1}^{s} f_k a_k R_r \\ &= \textstyle\sum_{i,j} \tau_i \langle v_{ij} \rangle R_r \oplus \sum_{\oplus k=1}^{s} f_k a_k R_r \, , \end{aligned} \tag{1}$$

where $f_1, \cdots, f_s \in \mathrm{Hom}_{S_l}(T, R)$, $a_1, \cdots, a_s \in \{e_{11}, \cdots, e_{nn}\}$, and $\oplus$ means a direct sum. Now let $\{d_1, \cdots, d_t\}$ $(t = [T : S]_l)$ be a left S-basis of T. Then there exists some $w_1, \cdots, w_t \in \mathrm{Hom}_{S_l}(T, R)$ such that $d_p w_p = 1$ and $d_p w_q = 0$ for $p \neq q$, and then we have

$$\mathrm{Hom}_{S_l}(T, R) = \textstyle\sum_{p=1}^{t} w_p R_r \, . \tag{2}$$

From (1) and (2), we can write $w_p = \sum_{i,j} \tau_i \langle v_{ij} \rangle x_{ijr}^{(p)} + \sum_{k=1}^{s} f_k a_k y_{kr}^{(p)}$, where $x_{ij}^{(p)}, y_k^{(p)} \in R$, $p = 1, \cdots, t$. Now, we choose a $T_2 \in \mathfrak{R}^0_{l.f}/\{T_1, T\tau_i \langle v_{ij} \rangle x_{ijr}^{(p)}, T f_k a_k y_{kr}^{(p)}, T \tau_i \langle v_{ij} \rangle, T f_k a_k, T\tau_i, T f_k, T, v_{ij}, x_{ij}^{(p)}, a_k, y_k^{(p)}, T w_p ;\ i = 1, \cdots, m, j = 1, \cdots, k = 1, \cdots, s, p = 1, \cdots, t\}$. Then there exists some $\sigma_2 \in \tilde{\mathfrak{S}}(T_2/S, R)$ such that $\sigma_2 | T_1 = \sigma_1$ (cf. [3, Lemma 12]). Since $d_p(w_p \sigma_2) = 1$ and $d_p(w_q \sigma_2) = 0$ for $p \neq q$, we have $\mathrm{Hom}_{S_l}(T, R) = \sum_{p=1}^{t} (w_p \sigma_2) R_r$. Then, from $w_p \sigma_2 = \sum_{i,j} \tau_i \sigma_2 \langle v_{ij} \sigma_2 \rangle (x_{ij}^{(p)} \sigma_2)_r + \sum_{k=1}^{s} (f_k \sigma_2)(a_k \sigma_2)(y_k^{(p)} \sigma_2)_r$, we obtain

$$\begin{aligned} \mathrm{Hom}_{S_l}(T, R) &= \textstyle\sum_{p=1}^{t} (w_p \sigma_2) R_r \\ &= \textstyle\sum_{i=1}^{m} \tau_i \sigma_2 \langle V \rangle R_r + \sum_{k=1}^{s} (f_k \sigma_2)(a_k \sigma_2) R_r . \end{aligned} \tag{3}$$

Clearly $[\sum_{k=1}^{s} (f_k \sigma_2)(a_k \sigma_2) R_r | R_r] \leqq [\sum_{k=1}^{s} f_k a_k R_r | R_r]$. Hence, by (1) and (3), we have $[\mathfrak{S}(T/S, R) R_r | R_r] \leqq [\sum_{i=1}^{m} \tau_i \sigma_2 \langle V \rangle R_r | R_r]$. Since $\mathfrak{S}(T/S, R) \supset \bigcup_{i=1}^{m} \tau_i \sigma_2 \langle V \rangle$, it follows that $[\mathfrak{S}(T/S, R) R_r | R_r] = [\sum_{i=1}^{m} \tau_i \sigma_2 \langle V \rangle R_r | R_r]$, which implies that $\mathfrak{S}(T/S, R) R_r = \sum_{i=1}^{m} \tau_i \sigma_2 \langle V \rangle R_r$. Then, by [3, Theorem 1], we obtain $\mathfrak{S}(T/S, R) = \bigcup_{i=1}^{m} \tau_i \sigma_2 \langle V \rangle$. From this, our assertion follows immediately.

Lemma 4. *Let R be left locally finite over S, and $S \in \mathfrak{R}$. If $J(\mathfrak{S}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l.f}$ then $(H \cap T) \tilde{\mathfrak{S}}(T/S, R) \subset H$ for every $T \in \mathfrak{R}^0_{l.f}$.*

Proof. For $T \in \mathfrak{R}^0_{l.f}$, we have $(\mathfrak{S}(T/S, R) : \langle V \rangle) < \infty$ by [3, Corollary 2 (i)], and we write $\mathfrak{S}(T/S, R) = \bigcup_{i=1}^{m} \tau_i \langle V \rangle$, where $m < \infty$, $\tau_i \in \mathfrak{S}(T/S, R)$, $\tau_1 = 1$. We set here $R^* = [T\tau_1, \cdots, T\tau_m, V]$ and $H^* = R^* \cap H$. Then $[T\tau_1, \cdots, T\tau_m] \in \mathfrak{R}^0_{l.f}$. Hence we have $[H^* : [S, C_0]]_l < \infty$ by Lemma 2, and we write $H^* = \sum_{\oplus j=1}^{n} [S, C_0] d_j$. Since $d_1, \cdots, d_n \in R^*$, there exists a finite

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[1] See [3, Remark 1]

subset E of V such that $d, \cdots, d_n \in [T\tau_1, \cdots, T\tau_m, E]$. We set $T_1 = [T\tau_1, \cdots, T\tau_m, E]$. Then $T_1 \in \mathfrak{R}^0_{l,f}$, $T_1 \subset R^*$, and $T_1 \mathfrak{G}(T_1/S, R) \subset R^*$. Now, let $\sigma \in \tilde{\mathfrak{G}}(T/S, R)$. Then, there exists some $\sigma_1 \in \tilde{\mathfrak{G}}(T_1/S, R)$ with $\sigma_1 | T = \sigma$, and then, by Lemma 3, there exists some $T_2 \in \mathfrak{R}^0_{l,f}/T_1$, $\tau \in \mathfrak{G}(T_1/S, R)$, $\sigma_2 \in \tilde{\mathfrak{G}}(T_2/S, R)$ such that $\tau\sigma_2 \sim 1$, $\sigma_2 | T_1 = \sigma_1$. From $\tau\sigma_2 \sim 1$, we have $\tau\sigma_2 | H \cap T_1 = 1$ (cf. [3, Remark 1]). First, we shall prove that

$$(1) \qquad (H \cap T_1)\tau \subset H^*.$$

Since $T_1 \mathfrak{G}(T_1/S, R) \subset R^*$, we have $(H \cap T_1)\tau \subset R^*$. If $h\tau \notin H^*$ for some $h \in H \cap T_1$ then $h\tau \notin H$. Hence there exists an element $u \in V$ such that $(h\tau)u \neq u(h\tau)$. For $T_2' \in \mathfrak{R}^0_{l,f}/[T_2, u]$, we can find some $\sigma_2' \in \tilde{\mathfrak{G}}(T_2'/S, R)$ with $\sigma_2' | T_2 = \sigma_2$. Then, we have that $((h\tau)u)\sigma_2' \neq (u(h\tau))\sigma_2'$, and so $h(u\sigma_2') \neq (u\sigma_2')h$. Hence $u\sigma_2' \notin V$. This is a contradiction. Now, we set $d_j\tau = f_j$, $j = 1, \cdots, n$. Then $f_j \in H^*$, $f_j\sigma_2 = d_j$ for $j = 1, \cdots, n$ in virtue of (1). Next, we shall prove that

$$(2) \qquad H^* = \textstyle\sum_{j=1}^{n} [S, C_0] f_j .$$

For an arbitrary finite subset F of C_0, and for $T_2^* \in \mathfrak{R}^0_{l,f}/[T_2, F]$, there exists a $\sigma_2^* \in \tilde{\mathfrak{G}}(T_2^*/S, R)$ such that $\sigma_2^* | T_2 = \sigma_2$. Then we have $F\sigma_2^* \subset C_0$ by Corollary 1. Hence $(\sum_{j=1}^{n} [S, F] f_j)\sigma_2^* = \sum_{j=1}^{n} [S, F\sigma_2^*](f_j\sigma_2^*) = \sum_{j-1}^{n} [S, F\sigma_2^*] d_j = \sum_{\oplus j=1}^{n} [S, F\sigma_2^*] d_j$. This implies that $\sum_{j=1}^{n} [S, F] f_j = \sum_{\oplus j=1}^{n} [S, F] f_j$, and hence $\sum_{j=1}^{n} [S, C_0] f_j = \sum_{\oplus j=1}^{n} [S, C_0] f_j$. Therefore it follows that $\sum_{j=1}^{n} [S, C_0] f_j = H^*$. Now, from (2) and $H \cap T \subset H^*$, we can find a finite subset F' of C_0 such that $H \cap T \subset \sum_{j=1}^{n} [S, F'] f_j$. For $T_3 \in \mathfrak{R}^0_{l,f}/[T_2, F']$, there exists a $\sigma_3 \in \tilde{\mathfrak{G}}(T_3/S, R)$ such that $\sigma_3 | T_2 = \sigma_2$. Then we have $(H \cap T)\sigma_3 \subset (\sum_{j=1}^{n} [S, F'] f_j)\sigma_3 \subset H^*$. Noting that $\sigma_3 | H \cap T = \sigma | H \cap T$, we obtain $(H \cap T)\sigma \subset H^* \subset H$. This completes the proof.

If R is $S \cdot V$-R-irreducible or R-$S \cdot V$-irreducible then $S \in \mathfrak{R}$ by [3, Lemma 1]. If $S \in \mathfrak{R}$ and R is left locally finite over S, and if $J(\mathfrak{G}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l,f}$, then $\tilde{\mathfrak{G}} = \{\tilde{\mathfrak{G}}(T/S, R) ; T \in \mathfrak{R}^0_{l,f}\}$ is a left q-system by Lemma 4 and [3, Lemma 14]. Hence [3, Theorem 5] takes the place of the next

Principal Theorem. *The following conditions* (A_l)——(C_r) *are all equivalent. Moreover,* R *is* q*-Galois and locally finite over* S *if and only if one of these conditions is satisfied.*

(A_l): (i) $J(\mathfrak{G}(T/S, R), T) = S$ for every $T \in \mathfrak{R}^0_{l,f}$, *and* R *is left locally finite over* S, *and*

(ii) R *is* $S \cdot V$-R*-irreducible.*

(B_l): $(A_l : \mathrm{i})$ *plus the condition that* R-$S \cdot V$*-irreducible.*

(C_l): $(A_l : \mathrm{i})$ *plus the condition that* $S, H \in \mathfrak{R}$ *and* $[V_R^2(T) : H]_l = [V :$

$V_R(T)]_r$ *for every* $T\in\mathfrak{R}_{l,f}^0$.

(A$_r$). (B$_r$). (C$_r$). (These are symmetric to (A$_l$)——(C$_l$)),

As a direct consequence of our theorem, we obtain the following

Corollary 2. *Let R be left locally finite over S. Then, R is q-Galois over S if and only if $J(\mathfrak{G}(S'/S, R), S')=S$ for every $S'\in\mathfrak{R}_{l,f}$, and R is $S\cdot V$-R-irreducible.*

Remark 2. Our theorem is a result that corresponds to [2, Theorem 1]. Let (α), (β), (γ), and (δ) be as in the difinifion of left q-systems given in [3]. If R is left locally finite over S and $S\in\mathfrak{R}$ then the last condition (δ) is a result of conditions (α), (β), and (γ), which can be proved by making use of the same method as in the proof of Lemma 4. Hence, in this case, a left q-system is defined as a system with (α), (β), and (γ).

Appendices.

In the rest of this note, we shall make a study of the S_l-S_r-homomorphism ring of R to see that our q-Galois extensions are also Elliger's quasi-Galois extensions [1].

Lemma 5. *Let R be q-Galois and locally finite over S. If $T\in\mathfrak{R}_{l,f}^0$ and M is a T_l-T_r-submodule of R which is finite over T, then*

$$\mathrm{Hom}_{S_l\cdot S_r}(M, R)=(\mathfrak{G}(T'/S, R)\,|\,M)V_r=(\mathfrak{G}(T'/S, R)\,|\,M)V_l$$

for every $T'\in\mathfrak{R}_{l,f}^0/M\cup T$.

Proof. We write $T=\sum_{i,j=1}^{m}A^*a_{ij}$, where a_{ij}'s are matrix units of T, and $A^*=V_T(\{a_{ij}\})$ which is a division ring. Then $V_R(\{a_{ij}\})=A$ is a division ring and $R=\sum_{i,j=1}^{n}Aa_{ij}$. Now we set $M^*=\{\sum_{i=1}^{n}a_{i1}xa_{1i}\,;\ x\in M\}$. Then $M^*=M\cap A$ and M^* is a finite A^*-module. If we write $M^*=\sum_{\oplus k=1}^{s}A^*d_k$ then we see that $M=\sum_{\oplus k=1}^{s}Td_k$. Hence M has a left S-basis, and so we have $\mathrm{Hom}_{S_l}(M, R)=\mathrm{Hom}_{S_l}(T', R)\,|\,M$ for every $T'\in\mathfrak{R}_{l,f}^0/M\cup T$. Therefore we obtain $\mathrm{Hom}_{S_l}(M, R)=\mathfrak{G}(T'/S, R)R_r\,|\,M=(\mathfrak{G}(T'/S, R)\,|\,M)R_r=\sum_{\oplus}(\sigma_i\,|\,M)R_r$, where $\sigma_i\in\mathfrak{G}(T'/S, R)$, and $(\sigma_i\,|\,M)R_r\cong R$ (as modules). From this it follows that $\mathrm{Hom}_{S_l\cdot S_r}(M, R)=\sum_{\oplus}(\sigma_i\,|\,M)V_r=(\mathfrak{G}(T'/S, R)\,|\,M)V_r$. Similarly, we have $\mathrm{Hom}_{S_l\cdot S_r}(M, R)=(\mathfrak{G}(T'/S, R)\,|\,M)V_l$.

Now we denote by $\mathfrak{T}$ the ring $\mathrm{Hom}_{S_l\cdot S_r}(R, R)$. Then we have the following

Proposition. *Let R be q-Galois and locally finite over S. Then*

$$\mathfrak{T}\,|\,T=\mathrm{Hom}_{S_l\cdot S_r}(T, R)=\mathfrak{G}(T/S, R)V_l=\mathfrak{G}(T/S, R)V_r$$

for every $T\in\mathfrak{R}_{l,f}^0$.

Proof. By Lemma 5, we have $\mathrm{Hom}_{S_l\cdot S_r}(T, R)=\mathfrak{G}(T/S, R)V_l=\mathfrak{G}(T/S, R)V_r$ $(T\in\mathfrak{R}^0_{l,r})$. Since $\mathfrak{T}\,|\,T\subset\mathrm{Hom}_{S_l\cdot S_r}(T, R)$ and $\mathfrak{T}$ is a V_l-V_r-module, it will suffice to show that $\mathfrak{G}(T/S, R)\subset\mathfrak{T}\,|\,T$. Let $\sigma\in\mathfrak{G}(T/S, R)$, and consider the set Φ of all pairs (M, f) consisting of a T_l-T_r-submodule M of R containing T, and an S_l-S_r-homomorphism f of M into R whose restriction to T is σ, Then, the set can be inductively ordered by prescribing $(M, f)\leqq(M', f')$ if $M\subset M'$ and $f'\,|\,M=f$. Hence we can apply Zorn's Lemma to find a maximal element (M_0, f_0) of Φ. Now, we assume that $M_0\neq R$. Then we can find some $T'\in\mathfrak{R}^0_{l,r}/T$ such that $T'\not\subset M_0$. Clearly, $M_0\cap T'$ is a T_l-T_r-module which is finite over T. Hence, by Lemma 5, we have that $f_0\,|\,M_0\cap T'\in\mathrm{Hom}_{S_l\cdot S_r}(M_0\cap T', R)=(\mathfrak{G}(T'/S, R)\,|\,M_0\cap T')V_r$, and so $f_0\,|\,M_0\cap T'=g\,|\,M_0\cap T'$ for some $g\in\mathfrak{G}(T'/S, R)V_r=\mathrm{Hom}_{S_l\cdot S_r}(T', R)$. We denote the map $h: M_0+T'\longrightarrow R$ by the following; for $a+b\in M_0+T'$ $(a\in M_0,\ b\in T')$,

$$(a+b)h=af_0+bg.$$

For a division subring F of S, we have

$$M_0+T'=\Sigma_{\oplus}Fx_i\oplus\Sigma_{\oplus}Fy_j\oplus\Sigma_{\oplus}Fz_k,$$

where $M_0=\Sigma_{\oplus}Fx_i\oplus\Sigma_{\oplus}Fy_j$, $M_0\cap T'=\Sigma_{\oplus}Fy_j$, $T'=\Sigma_{\oplus}Fy_j\oplus\Sigma_{\oplus}Fz_k$. From this, we see that h is a single-valued map. Moreover, one will easily see that h is an S_l-S_r-homomorphism. Clearly $h\,|\,T=f_0\,|\,T=\sigma$. Hence $(M_0+T', h)\in\Phi$ and $(M_0, f_0).<(M_0+T', h)$. This is a contradiction. Therefore we obtain $\mathfrak{G}(T/S, R)\subset\mathfrak{T}\,|\,T$.

From Proposition, we obtain the following

Corollary 3. *Let R be q-Galois and locally finite over S. Then*

i) $\mathrm{Hom}_{\mathfrak{T}}(R, R)\cap R_l=S_l$,

ii) $\mathrm{Hom}_{\mathfrak{T}}(R, R)\cap R_r=S_r$,

iii) $\mathfrak{T}R_r$ *is dense in* $\mathrm{Hom}_{S_l}(R, R)$,

iv) $\mathfrak{T}R_l$ *is dense in* $\mathrm{Hom}_{S_r}(R, R)$.

Remark 3. By Corollary 3, $\mathfrak{T}R_r$ is dense in the ring $[\mathfrak{T}, R_r]$ generated by $\mathfrak{T}$ and R_r. Similarly, $\mathfrak{T}R_l$ is dense in the ring $[\mathfrak{T}, R_l]$ generated by $\mathfrak{T}$ and R_l.

In [1], S. Elliger has defined quasi-Galois extensions of division rings as follows: If for a division ring extension K/L there exists a multiplicative semi-group $\mathfrak{H}$ of L_l-L_r-homomorphisms of K such that $\mathrm{Hom}_{\mathfrak{H}}(K, K)\cap K_l=L_l$ and $\mathrm{Hom}_{\mathfrak{H}}(K, K)\cap K_r=L_r$, then K is called quasi-Galois over L. By Corollary 3, our q-Galois extensions are also Elliger's quasi-Galois extensions. Any purely inseparable extension (commutative field) is Elliger's quasi-Galois but not our q-Galois,

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DEPARTMENT of MATHEMATICS,
OKAYAMA UNIVERSITY.

(*Received June 10, 1966*)