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Abstract

There is an open question whether Fuchsian groups having the same set of the axes of hyperbolic elements are commensurable or not. In this note, we consider an analogous question where the axes are replaced with the fixed points of parabolic elements.

KEYWORDS: Fuchsian group, arithmetic, commensurable

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1. Introduction

For a compact Riemannian manifold M, the spectrum of the Laplacian of M is defined to be the set of all eigenvalues of the Laplace-Beltrami operator on $L^2(M)$. Two compact Riemannian manifolds are called isospectral if the spectra of their Laplacians are the same. It is a classical problem whether isospectrality implies isometricity or not. Two isometric manifolds are obviously isospectral but the inverse statement does not necessarily hold. Since the first example of an isospectral but non-isometric manifold is constructed by Milnor [9], a lot of examples of such manifolds have been reported (see [2], [3] and [12]).

Those examples have a common property: isospectral but non-isometric manifolds are commensurable. Here two compact Riemannian manifolds are called commensurable if they have a common finite cover. Then there arises a new problem whether isospectrality implies commensurability. As an answer to this problem in a special case, Reid [11] proved that, if arithmetic hyperbolic 2- or 3-manifolds are isospectral, then they are commensurable.

On the other hand, for a compact Riemannian manifold M, the length spectrum of M is defined to be the set of all lengths of closed geodesics in M. Two compact Riemannian manifolds are called iso-length-spectral if their length spectra are the same. Hyperbolic 2-manifolds are isospectral if and only if they are iso-length-spectral (see [8]). While the length spectrum extracts only the information on their lengths from the closed geodesics, we can also consider the distribution of the axes of hyperbolic elements of the Fuchsian group Γ with $\mathbf{H}/\Gamma \cong M$ since they correspond to closed geodesics in M. As before, we expect that, if two Fuchsian groups are iso-axial, then they should be commensurable. Concerning this problem, the following theorem has been also proved by Long and Reid [4].

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Theorem 1.1. *Let* Γ_1 *and* Γ_2 *be arithmetic Fuchsian groups. If* $\mathrm{Ax}(\Gamma_1) = \mathrm{Ax}(\Gamma_2)$ *then* Γ_1 *and* Γ_2 *are commensurable.*

Here $\mathrm{ax}(\gamma) \subset \mathbf{H}$ denotes the axis of a hyperbolic element $\gamma \in PSL(2,\mathbf{R})$, and for a subgroup $\Gamma \subset PSL(2,\mathbf{R})$, the set $\mathrm{Ax}(\Gamma)$ is defined to be

$$\mathrm{Ax}(\Gamma) = \{\mathrm{ax}(\gamma) \mid \gamma \in \Gamma \text{ is hyperbolic}\}.$$

For subgroups G_1 and G_2 in a group G, we call G_1 and G_2 are commensurable if $G_1 \cap G_2$ has a finite index both in G_1 and G_2.

In this paper, we prove the following analogous theorem.

Theorem 1.2. *Let* Γ_1 *and* Γ_2 *be arithmetic Fuchsian groups having parabolic elements. If* $\mathrm{Fx_p}(\Gamma_1) = \mathrm{Fx_p}(\Gamma_2)$ *then* Γ_1 *and* Γ_2 *are commensurable.*

Here $\mathrm{fix}(\gamma) \subset \overline{\mathbf{H}}$ denotes the set of fixed points of an element $\gamma \in PSL(2,\mathbf{R})$, and for a subgroup $\Gamma \subset PSL(2,\mathbf{R})$, the set $\mathrm{Fx_p}(\Gamma)$ is defined to be

$$\mathrm{Fx_p}(\Gamma) = \{\mathrm{fix}(\gamma) \mid \gamma \in \Gamma \text{ is parabolic}\}.$$

Actually, the converse of both Theorems 1.1 and 1.2 are also true without the assumption of arithmeticity. Indeed, if Γ_1 and Γ_2 in $PSL(2,\mathbf{R})$ are commensurable, then, for any hyperbolic or parabolic element γ in Γ_i $(i = 1, 2)$, there exists $n \in \mathbf{N}$ such that γ^n is contained in $\Gamma_1 \cap \Gamma_2$ and $\gamma^n \neq id$. Thus we obtain $\mathrm{Ax}(\Gamma_1) = \mathrm{Ax}(\Gamma_1 \cap \Gamma_2) = \mathrm{Ax}(\Gamma_2)$ and $\mathrm{Fx_p}(\Gamma_1) = \mathrm{Fx_p}(\Gamma_1 \cap \Gamma_2) = \mathrm{Fx_p}(\Gamma_2)$.

On the other hand, for the fixed point set

$$\mathrm{Fx_e}(\Gamma) = \{\mathrm{fix}(\gamma) \mid \gamma \in \Gamma \text{ is elliptic}\},$$

we have already proved the following theorem in our previous paper [7]. It should be noted that we do not have to assume the arithmeticity of Fuchsian groups in this theorem. However the converse does not necessarily hold differently from the previous theorems.

Theorem 1.3. *Let* Γ_1 *and* Γ_2 *be cofinite Fuchsian groups having elliptic elements. If* $\mathrm{Fx_e}(\Gamma_1) = \mathrm{Fx_e}(\Gamma_2)$ *then* Γ_1 *and* Γ_2 *are commensurable.*

We have not known yet if Theorem 1.1 can be extended to cofinite Fuchsian groups. However, Theorem 1.2 cannot be extended to cofinite Fuchsian groups, as Long and Reid [5] proved that there exist non-arithmetic, cofinite Fuchsian groups Γ_1 and Γ_2 that are not commensurable but satisfies $\mathrm{Fx_p}(\Gamma_1) = \mathrm{Fx_p}(\Gamma_2)$.

The proof of Theorem 1.2 basically traces that of Theorem 1.1 given in [4], but a new contribution can be found in the proof of the following theorem, which also has the corresponding result in [4]. Let $\mathrm{Comm}(\Gamma)$ be the commensurator of Γ, a group of the elements $\gamma \in PSL(2,\mathbf{R})$ for which

$\gamma\Gamma\gamma^{-1}$ is commensurable with Γ, and $\Sigma_p(\Gamma)$ the set of all $\gamma \in PSL(2,\mathbf{R})$ that preserve $\mathrm{Fx_p}(\Gamma)$.

Theorem 1.4. *Let Γ be an arithmetic Fuchsian group. Then* $\mathrm{Comm}(\Gamma) = \Sigma_p(\Gamma)$.

We will prove Theorem 1.4 in Sections 3 and 4, and Theorem 1.2 in Section 5.

2. Preliminaries

In this section we define arithmetic Fuchsian groups.

2.1. **Quaternion algebras.** First we define quaternion algebras. Let k be a field. Let A be an algebra over k whose dimension as a vector space over k is four. If A has a basis $\{1, i, j, l\}$ that satisfies the following conditions, then we call A a quaternion algebra.

(1) 1 is a multiplicative identity element of A.
(2) There exists $a, b \in k$ such that $i^2 = a1$ and $j^2 = b1$.
(3) $ij = -ji = l$.

In this case, we write $A = (\frac{a,b}{k})$. This notation is called the Hilbert symbol. It should be noted that a quaternion algebra does not uniquely determine the Hilbert symbol.

Let A be a quaternion algebra over a field k. Let $\{1, i, j, l\}$ be a basis that satisfies conditions (1), (2) and (3). For an element $x = x_0 + x_1 i + x_2 j + x_3 l \in A$, the reduced norm $n_A(x)$ is defined by $x_0^2 - ax_1^2 - bx_2^2 + abx_3^2$. It is known that this definition is independent of a choice of the basis (see [6] p.80).

A trivial example of a quaternion algebra is Hamilton's quaternion field $\mathcal{H}$. We can write $\mathcal{H} = (\frac{-1,-1}{\mathbf{R}})$. Another example is $M(2,\mathbf{R}) = (\frac{1,1}{\mathbf{R}})$, for which

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$$

is a basis that satisfies (1), (2) and (3). We obtain that $n_{M(2,\mathbf{R})}(x) = \det x$ for $x \in M(2,\mathbf{R})$ by computation.

It is a well-known fact that a quaternion algebra is a simple central algebra (see [6] p.78). Then the Skolem-Noether Theorem tells us that an automorphism of any quaternion algebra is an inner automorphism (see [6] p.107).

2.2. **Definition of arithmetic Fuchsian groups.** Here we define arithmetic Fuchsian groups. First we define an order of a quaternion algebra. Let k be a number field and A be a quaternion algebra over k. Let R_k be the set of all algebraic integers in k. We can regard A as an R_k-module. If a subset $\mathcal{O}$ of A satisfies the following conditions, then we call $\mathcal{O}$ an order.

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(1) $\mathcal{O}$ is a finitely generated R_k-module.
(2) $\mathcal{O}$ is a subring containing 1.
(3) $\mathcal{O} \otimes_{R_k} k \cong A$.

Let k be a totally real number field, that is to say, if α is an algebraic number such that $k = \mathbf{Q}(\alpha)$ then, for all roots $\alpha_1 = \alpha, \alpha_2, \alpha_3, \ldots, \alpha_n$ of the minimal polynomial of α, $\mathbf{Q}(\alpha_i)$ $(i = 1, 2, \ldots, n)$ is contained in $\mathbf{R}$. Let σ_i be the isomorphism of k into $\mathbf{Q}(\alpha_i)$ such that $\sigma_i(\alpha) = \alpha_i$ $(\sigma_1 = id)$. Let A be a quaternion algebra over k that has the Hilbert symbol $(\frac{a,b}{k})$ whose scalar a and b satisfy the following conditions.

(1) There exists a basis $\{1, i, j, l\}$ of $M(2, \mathbf{R})$ with
- 1 is a multiplicative identity element of $M(2, \mathbf{R})$;
- $i^2 = \sigma_1(a) = a$;
- $j^2 = \sigma_1(b) = b$;
- $ij = -ji = l$.

(2) For each $m = 2, 3, \ldots, n$, there exists a basis $\{1, i_m, j_m, l_m\}$ of $\mathcal{H}$ with
- 1 is a multiplicative identity element of $\mathcal{H}$;
- $i_m^2 = \sigma_m(a)$;
- $j_m^2 = \sigma_m(b)$;
- $i_m j_m = -j_m i_m = l_m$.

From (1), we know that $A \otimes_k \mathbf{R} \cong M(2, \mathbf{R})$.

Let $\mathcal{O}$ be an order of A and define $\mathcal{O}^1$ to be

$$\mathcal{O}^1 = \{x \in \mathcal{O} \mid n_A(x) = 1\}.$$

Let ρ be a k-embedding of A into $M(2, \mathbf{R})$. Let p be the canonical projection from $SL(2, \mathbf{R})$ to $PSL(2, \mathbf{R})$. Then it is known that $\rho(\mathcal{O}^1)$ is discrete in $SL(2, \mathbf{R})$ and $p\rho(\mathcal{O}^1)$ is a cofinite Fuchsian group (see [6] p.259). A Fuchsian group Γ is called arithmetic if Γ is commensurable with some such $p\rho(\mathcal{O}^1)$.

2.3. **Invariant quaternion algebras.** We provide useful tools for considering arithmetic Fuchsian groups and state certain propositions. For a non-elementary subgroup Γ of $PSL(2, \mathbf{R})$, we define $\Gamma^{(2)}$, $k\Gamma$ and $A\Gamma$ to be

$$\begin{aligned} \Gamma^{(2)} &= \langle \gamma^2 \mid \gamma \in \Gamma \rangle; \\ k\Gamma &= \mathbf{Q}\,(\mathrm{tr}\,\gamma \mid \gamma \in p^{-1}(\Gamma^{(2)})); \\ A\Gamma &= \{\sum_{\text{finite}} a_i \gamma_i \mid a_i \in k\Gamma,\ \gamma_i \in p^{-1}(\Gamma^{(2)})\}. \end{aligned}$$

By definition, $A\Gamma$ is a $k\Gamma$-algebra contained in $M(2, \mathbf{R})$. In fact, it is a $k\Gamma$-quaternion algebra (see [6] p.114). It is known that the pair $(k\Gamma, A\Gamma)$ is an invariant for a commensurability class of Γ, and thus $A\Gamma$ is called an invariant quaternion algebra.

For arithmetic Fuchsian groups, the following three propositions are known (see [6] p.268, p.265 and p.270 respectively).

Proposition 2.1. *Let Γ_1 and Γ_2 be arithmetic Fuchsian groups. Then Γ_1 and Γ_2 are commensurable if and only if $(k\Gamma_1, A\Gamma_1) = (k\Gamma_2, A\Gamma_2)$.*

Proposition 2.2. *Let Γ be an arithmetic Fuchsian group that is commensurable with $p\rho(\mathcal{O}^1)$, where $\mathcal{O}$ is an order of a quaternion algebra A over a field k and ρ is a k-embedding of A into $M(2,\mathbf{R})$. Then $k = k\Gamma$ and $\rho(A) = A\Gamma$.*

Thus, if Γ is arithmetic, then there exists an order $\mathcal{O}_{A\Gamma}$ of $A\Gamma$ such that Γ is commensurable with $p(\mathcal{O}^1_{A\Gamma})$.

If subgroups Γ_1 and Γ_2 of $PSL(2,\mathbf{R})$ are commensurable, then we write $\Gamma_1 \sim \Gamma_2$. Note that $\sim$ is an equivalence relation. We define the commensurator of Γ as

$$\mathrm{Comm}(\Gamma) = \{\gamma \in PSL(2,\mathbf{R}) \mid \gamma\Gamma\gamma^{-1} \sim \Gamma\}.$$

Let P be the canonical projection of $GL_+(2,\mathbf{R})$ onto $PGL_+(2,\mathbf{R})$ with the correspondence $x \mapsto [x]$. Let ψ be a homomorphism of $GL_+(2,\mathbf{R})$ onto $SL(2,\mathbf{R})$ defined by $\psi(x) = \frac{x}{\sqrt{\det x}}$ for $x \in GL_+(2,\mathbf{R})$, and φ an isomorphism of $PGL_+(2,\mathbf{R})$ onto $PSL(2,\mathbf{R})$ sending $[x]$ to $[\frac{x}{\sqrt{\det x}}]$. Then p, P, φ and ψ satisfy the following commutative diagram.

$$\begin{array}{ccc} GL_+(2,\mathbf{R}) & \xrightarrow{P} & PGL_+(2,\mathbf{R}) \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ SL(2,\mathbf{R}) & \xrightarrow{p} & PSL(2,\mathbf{R}) \end{array}$$

Proposition 2.3. *If Γ is an arithmetic Fuchsian group, then* $\mathrm{Comm}(\Gamma) = \varphi P(A\Gamma_+)$, *where*

$$A\Gamma_+ = \{x \in A\Gamma \mid n_{A\Gamma}(x) = \det x > 0\}.$$

3. Proof of Theorem 1.4

In this section we prove Theorem 1.4 modulo a certain claim which will be shown in the next section. We define

$$\Sigma_p(\Gamma) = \{\gamma \in PSL(2,\mathbf{R}) \mid \gamma(\mathrm{Fx_p}(\Gamma)) = \mathrm{Fx_p}(\Gamma)\}.$$

Theorem 1.4. *Let Γ be an arithmetic Fuchsian group. Then* $\mathrm{Comm}(\Gamma) = \Sigma_p(\Gamma)$.

Proof. First we show the inclusion $\mathrm{Comm}(\Gamma) \subset \Sigma_p(\Gamma)$, which is easier than the other direction. In fact, this inclusion holds even for a non-arithmetic Fuchsian group Γ. Let γ be an element of $\mathrm{Comm}(\Gamma)$ and g a parabolic element of Γ. Because $\gamma\Gamma\gamma^{-1}$ is commensurable with Γ, we obtain $\mathrm{Fx}_\mathrm{p}(\Gamma) = \mathrm{Fx}_\mathrm{p}(\gamma\Gamma\gamma^{-1})$. Thus

$$\gamma(\mathrm{fix}(g)) = \mathrm{fix}(\gamma g \gamma^{-1}) \in \mathrm{Fx}_\mathrm{p}(\gamma\Gamma\gamma^{-1}) = \mathrm{Fx}_\mathrm{p}(\Gamma).$$

It follows that $\mathrm{Comm}(\Gamma) \subset \Sigma_p(\Gamma)$.

Next we prove the inclusion $\mathrm{Comm}(\Gamma) \supset \Sigma_p(\Gamma)$. Let v be an element of $\Sigma_p(\Gamma)$ and V an element of $SL(2, \mathbf{R})$ such that $p(V) = v$. We will show that there exist $r \in \mathbf{R}$ and $W \in A\Gamma_+$ such that $V = rW$. Then we obtain that

$$v = p(V) = p\psi(V) = \varphi P(rW) = \varphi P(W) \in \varphi P(A\Gamma_+),$$

from which $\Sigma_p(\Gamma) \subset \varphi P(A\Gamma_+) = \mathrm{Comm}(\Gamma)$ follows.

Let $\Phi : A\Gamma \to M(2, \mathbf{R})$ be the conjugation by V, that is, $\Phi(X) = VXV^{-1}$ for any element $X \in A\Gamma$. Then we will see $\Phi(A\Gamma) \subset A\Gamma$, which means that Φ is an automorphism of $A\Gamma$. This is the essential step in our arguments and the proof for this fact will be given in the next section. As noted in Section 2.1, Φ is actually an inner automorphism. Then there exists $W \in A\Gamma^* = \{x \in A\Gamma \mid \det x \neq 0\}$ that satisfies $VXV^{-1} = WXW^{-1}$ for every element $X \in A\Gamma$.

Thus $W^{-1}V$ is commutable with every element of $A\Gamma$. Although $A\Gamma$ is central, this does not necessarily imply that $W^{-1}V$ is contained in $(k\Gamma)I$. However, there exists a quaternion algebra A over a number field k such that $A \otimes_k \mathbf{R} \cong M(2, \mathbf{R})$ because Γ is arithmetic. Since $k = k\Gamma$ and $A \cong A\Gamma$ by Proposition 2.2, we know that $A\Gamma \otimes_{k\Gamma} \mathbf{R} \cong M(2, \mathbf{R})$. Then $W^{-1}V$ is commutable with every element of $M(2, \mathbf{R})$. Thus there exists $r \in \mathbf{R}$ such that $W^{-1}V = rI$, which shows that $V = rW$. From this we know that $\det W = \frac{1}{r^2} > 0$ hence $W \in A\Gamma_+$. This complete the proof. ☐

4. Φ preserves $A\Gamma$

To complete the proof of Theorem 1.4, we prove $\Phi(A\Gamma) \subset A\Gamma$. We will show that there exists a basis $\{X_1, X_2, X_3, X_4\}$ of $A\Gamma$ such that $\Phi(X_i) = VX_iV^{-1}$ $(i = 1, 2, 3, 4)$ is contained in $A\Gamma$. To this end, we prepare the following two lemmas.

Lemma 4.1. *Let Γ be a non-elementary subgroup of $PSL(2, \mathbf{R})$. If $X, Y \in A\Gamma_+$ satisfy $\mathrm{tr}(\psi(XYX^{-1}Y^{-1})) \neq 2$, then $\{I, X, Y, XY\}$ is a basis of $A\Gamma$ over $k\Gamma$.*

Proof. Because $A\Gamma$ is four dimensional over $k\Gamma$, it suffices to show that $\{I, X, Y, XY\}$ are linearly independent over $\mathbf{R}$. This is equivalent to saying that $\{I, \psi(X), \psi(Y), \psi(XY)\}$ are linearly independent over $\mathbf{R}$.

For any elements $A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}$ and $B = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4 \end{pmatrix}$ in $SL(2, \mathbf{R})$, set $C = \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix}$ to be $C = AB$. Then we have

$$\det \begin{pmatrix} 1 & a_1 & b_1 & c_1 \\ 0 & a_2 & b_2 & c_2 \\ 0 & a_3 & b_3 & c_3 \\ 1 & a_4 & b_4 & c_4 \end{pmatrix} = 2 - \mathrm{tr} ABA^{-1}B^{-1}.$$

Thus $\{I, A, B, AB\}$ are linearly independent over $\mathbf{R}$ if and only if

$$\mathrm{tr} ABA^{-1}B^{-1} \neq 2.$$

This shows that if $\mathrm{tr}(\psi(XYX^{-1}Y^{-1})) \neq 2$, then $\{I, \psi(X), \psi(Y), \psi(XY)\}$ are linearly independent over $\mathbf{R}$. □

Lemma 4.2. *Let A, B, A' and B' be elements of $SL(2, \mathbf{R})$ and define a, b, a' and b' to be $a = p(A), b = p(B), a' = p(A')$ and $b' = p(B')$ respectively. If a, b, a' and b' are parabolic elements of $PSL(2, \mathbf{R})$ and satisfy*

$$\mathrm{fix}(a) = \mathrm{fix}(a'); \quad \mathrm{fix}(b) = \mathrm{fix}(b'),$$

then there exists $r \in \mathbf{R}$ such that $AB - BA = r(A'B' - B'A')$.

Proof. Set $z = \mathrm{fix}(a)$ and $w = \mathrm{fix}(b)$, which are in $\mathbf{R} \cup \{\infty\}$. Because a and b are parabolic, it is known that (see [10] p.4), if $z \neq \infty$ and $w \neq \infty$, there exist r and s in $\mathbf{R} - \{0\}$ such that

$$A = \begin{pmatrix} 1 + rz & -rz^2 \\ r & 1 - rz \end{pmatrix}; \quad B = \begin{pmatrix} 1 + sw & -sw^2 \\ s & 1 - sw \end{pmatrix}.$$

Similarly, if $z = \infty$ and $w = \infty$, there exist t and u in $\mathbf{R} - \{0\}$ such that

$$A = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}.$$

Thus, if $z \neq \infty$ and $w \neq \infty$, then

$$AB - BA = rs \begin{pmatrix} w^2 - z^2 & 2z^2w - 2zw^2 \\ 2w - 2z & -w^2 + z^2 \end{pmatrix};$$

if $z \neq \infty$ and $w = \infty$, then

$$AB - BA = st \begin{pmatrix} 1 & -2w \\ 0 & -1 \end{pmatrix};$$

and if $z = \infty$ and $w = \infty$, then

$$AB - BA = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

The same results holds for a' and b' because $\mathrm{fix}(a) = \mathrm{fix}(a')$ and $\mathrm{fix}(b) = \mathrm{fix}(b')$. This shows the assertion. □

We begin the proof of the existence of a basis $\{X_1, X_2, X_3, X_4\}$ of $A\Gamma$ such that $\Phi(X_i) \in A\Gamma$. Let $\mathcal{O}$ be an order of $A\Gamma$ such that $p(\mathcal{O}^1) \sim \Gamma$ (we have mentioned the existence of such $\mathcal{O}$ in Section 2.3). Let $\widetilde{\mathscr{I}}$ denote the set of all pairs $(A, B) \in \mathcal{O}^1 \times \mathcal{O}^1$ such that $p(A)$ and $p(B)$ are parabolic elements of $\Gamma \cap p(\mathcal{O}^1)$ having disjoint fixed point sets:

$$\begin{aligned} \widetilde{\mathscr{I}} \;=\; & \{(A, B) \in \mathcal{O}^1 \times \mathcal{O}^1 \mid \\ & \text{parabolic } p(A),\ p(B) \in \Gamma \cap p(\mathcal{O}^1),\ \mathrm{fix}(p(A)) \cap \mathrm{fix}(p(B)) = \emptyset\}, \end{aligned}$$

and set

$$\mathscr{I} = \{AB - BA \mid (A, B) \in \widetilde{\mathscr{I}},\ \det(AB - BA) > 0\}.$$

Note that $\mathscr{I}$ is contained in $\mathrm{A}\Gamma_+$. Let $\langle \mathscr{I} \rangle$ be the subgroup of $GL_+(2, \mathbf{R})$ generated by $\mathscr{I}$. We will show that there exists a basis $\{X_1, X_2, X_3, X_4\}$ of $A\Gamma$ in $\langle \mathscr{I} \rangle$ such that $\Phi(X_i) = VX_iV^{-1}$ $(i = 1, 2, 3, 4)$ is contained in $A\Gamma$.

First we prove that $\varphi P(\langle \mathscr{I} \rangle)$ is a normal subgroup of $\varphi P(A\Gamma_+)$ and obtain that it is non-elementary because it is a normal subgroup of non-elementary $\varphi P(A\Gamma_+) = \mathrm{Comm}(\Gamma)$ $(\supset \Gamma)$. Let X be an element of $A\Gamma_+$ and $AB - BA$ an element of $\mathscr{I}$. Define x, a and b to be $x = \varphi P(X)$, $a = \varphi P(A) = p(A)$ and $b = \varphi P(B) = p(B)$. Note that a and b are parabolic elements of Γ. Because x is an element of $\varphi P(A\Gamma_+) = \mathrm{Comm}(\Gamma) \subset \Sigma_p(\Gamma)$, there exist parabolic elements a' and b' in Γ such that

$$x(\mathrm{fix}(a)) = \mathrm{fix}(xax^{-1}) = \mathrm{fix}(a'); \quad x(\mathrm{fix}(b)) = \mathrm{fix}(xbx^{-1}) = \mathrm{fix}(b').$$

Here a' and b' can be taken from $\Gamma \cap p(\mathcal{O}^1)$ because $\Gamma \cap p(\mathcal{O}^1)$ has a finite index in Γ. In other words, there exists $(A', B') \in \widetilde{\mathscr{I}}$ such that $p(A') = a'$ and $p(B') = b'$. By Lemma 4.2, we have

$$XAX^{-1}XBX^{-1} - XBX^{-1}XAX^{-1} = r(A'B' - B'A')$$

for some $r \in \mathbf{R}$. From this equation, we know that $\det(A'B' - B'A') = \frac{1}{r^2}\det(AB - BA) > 0$. Then we obtain that

$$\begin{aligned} & \varphi P(X)\varphi P(AB - BA)\varphi P(X)^{-1} \\ = \;& \varphi P(XAX^{-1}XBX^{-1} - XBX^{-1}XAX^{-1}) \\ = \;& \varphi P(r(A'B' - B'A')) \\ = \;& \varphi P(A'B' - B'A') \in \varphi P(\langle \mathscr{I} \rangle). \end{aligned}$$

This shows that $\varphi P(\langle \mathscr{I} \rangle)$ is a normal subgroup of $\varphi P(A\Gamma_+)$.

Next we will show that there exist a basis of $A\Gamma$ in $\langle\mathscr{I}\rangle$. By Lemma 4.1, we have only to show that there exist X and Y in $\langle\mathscr{I}\rangle$ that satisfy $\mathrm{tr}(\psi(XYX^{-1}Y^{-1})) \neq 2$. Suppose that X and Y in $\langle\mathscr{I}\rangle$ satisfy

$$\mathrm{tr}(\psi(XYX^{-1}Y^{-1})) = 2.$$

Then $p\psi(XYX^{-1}Y^{-1})$ is parabolic and there exist Z in $\langle\mathscr{I}\rangle$ such that

$$\mathrm{fix}(p\psi(Z)) \cap \mathrm{fix}(p\psi(XYX^{-1}Y^{-1})) = \emptyset$$

because $\varphi P(\langle\mathscr{I}\rangle) = p\psi(\langle\mathscr{I}\rangle)$ is non-elementary. Thus $X' := XYX^{-1}Y^{-1}$ and $Y' := Z(XYX^{-1}Y^{-1})Z^{-1}$ in $\langle\mathscr{I}\rangle$ satisfy $\mathrm{fix}(p\psi(X')) \cap \mathrm{fix}(p\psi(Y')) = \emptyset$ and $p\psi(X')$ is parabolic. Hence we obtain that $\mathrm{tr}(\psi(X'Y'X'^{-1}Y'^{-1})) \neq 2$ because if A and B in $SL(2,\mathbf{R})$ satisfy $\mathrm{fix}(p(A)) \cap \mathrm{fix}(p(B)) = \emptyset$ and $p(A)$ is parabolic, then $\mathrm{tr}(ABA^{-1}B^{-1}) \neq 2$ (see [1] p.185). This shows the assertion.

Now we have obtained X and Y in $\langle\mathscr{I}\rangle$ such that $\{I, X, Y, XY\}$ is a basis of $A\Gamma$. Then we will show that $\Phi(X) = VXV^{-1}$ and $\Phi(Y) = VYV^{-1}$ are contained in $A\Gamma$. Since X and Y are elements of $\langle\mathscr{I}\rangle$, there exist $(A_1, B_1), \ldots, (A_n, B_n)$ and $(C_1, D_1), \ldots, (C_m, D_m)$ in $\widetilde{\mathscr{I}}$ such that

$$X = \prod_{i=1}^{n}(A_iB_i - B_iA_i); \quad Y = \prod_{j=1}^{m}(C_jD_j - D_jC_j).$$

Then we have only to show that $V(\mathrm{A}_iB_i - B_iA_i)V^{-1}$ is contained in $A\Gamma$. Because v is an element of $\Sigma_p(\Gamma)$, there exists $(A'_i, B'_i) \in \widetilde{\mathscr{I}}$ such that

$$v(\mathrm{fix}(p(A_i))) = \mathrm{fix}(p(VA_iV^{-1})) = \mathrm{fix}(p(A'_i))$$

and

$$v(\mathrm{fix}(p(B_i))) = \mathrm{fix}(p(VB_iV^{-1})) = \mathrm{fix}(p(B'_i)).$$

By Lemma 4.2, there exists $r \in \mathbf{R}$ such that

$$V(A_iB_i - B_iA_i)V^{-1} = r(A'_iB'_i - B'_iA'_i).$$

This shows that

$$r = \frac{\mathrm{tr}(A_iB_i - B_iA_i)}{\mathrm{tr}(A'_iB'_i - B'_iA'_i)}$$

is contained in $k\Gamma$ because $\mathrm{tr}(A\Gamma) \subset k\Gamma$. Thus $V(A_iB_i - B_iA_i)V^{-1}$ is contained in $A\Gamma$, and hence $\Phi(A\Gamma) \subset A\Gamma$. This completes the proof.

5. Proof of the main theorem

Finally we prove Theorem 1.2.

Theorem 1.2. *Let* Γ_1 *and* Γ_2 *be arithmetic Fuchsian groups. If* $\mathrm{Fx_p}(\Gamma_1) = \mathrm{Fx_p}(\Gamma_2)$ *then* Γ_1 *and* Γ_2 *are commensurable.*

Proof. It is clear from definition that $\Gamma_1 \subset \Sigma_p(\Gamma_1)$. On the other hand, it follows that $\Sigma_p(\Gamma_1) = \Sigma_p(\Gamma_2)$ because $\mathrm{Fx_p}(\Gamma_1) = \mathrm{Fx_p}(\Gamma_2)$. Then we obtain that

$$\Gamma_1 \subset \Sigma_p(\Gamma_2) = \mathrm{Comm}(\Gamma_2) = \varphi P((A\Gamma_2)_+)$$

by Theorem 1.4 and Proposition 2.3. This shows that any element $x \in \Gamma_1$ can be written as $x = \varphi P(X)$ for some $X \in (A\Gamma_2)_+$. We have

$$x^2 = \varphi P(X^2) = p\psi(X^2) = p\left(\frac{X^2}{\det X}\right)$$

and hence $p^{-1}(x^2) = \{\frac{X^2}{\det X}, \frac{-X^2}{\det X}\}$. This implies that $\mathrm{tr}(p^{-1}(x^2)) \subset k\Gamma_2$ because $\det X = n_{A\Gamma_2}(X)$ is contained in $k\Gamma_2$.

From the above facts, we have $k\Gamma_1 = \mathbf{Q}(\mathrm{tr}(p^{-1}(\Gamma_1^{(2)})) \subset k\Gamma_2$ as well as

$$A\Gamma_1 = \{\sum_{\text{finite}} a_i\gamma_i \mid a_i \in k\Gamma_1, \gamma_i \in p^{-1}(\Gamma_1^{(2)})\} \subset A\Gamma_2.$$

Exchanging the roles of Γ_1 and Γ_2, we have $k\Gamma_1 = k\Gamma_2$ and $A\Gamma_1 = A\Gamma_2$. Thus, by Proposition 2.1, we see that Γ_1 is commensurable with Γ_2. $\square$

Remark. Generalizing arithmetic Fuchsian groups, we consider arithmetic Kleinian groups (see [6]). We can also extend Theorem 1.2 (and Theorem 1.1 as is mentioned in [4]) in the same way and obtain that, if $\mathrm{Fx_p}(\Gamma_1) = \mathrm{Fx_p}(\Gamma_2)$ for arithmetic Kleinian groups Γ_1 and Γ_2, then Γ_1 and Γ_2 are commensurable.

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