Primary ideal representations in non-commutative rings

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PRIMARY IDEAL REPRESENTATIONS IN 
NON-COMMUTATIVE RINGS

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Introduction. In his paper [4], H. Tominaga has given a necessary 
and sufficient condition that every ideal in a (non-commutative) ring 
be represented as the intersection of a finite number of s-right and s- 
left primary ideals\(^1\). It is the purpose of this paper to present a condition 
that every ideal in a ring be represented as a finite intersection of s-right 
primary ideals. After several definitions (§ 1), we shall prove in § 2 the 
uniqueness theorem for s-right primary representations: in any two s-right 
primary short representations of an ideal, the number of s-right primary 
components are the same and their radicals coincide in some order. In § 3, 
we shall give a necessary and sufficient condition that every ideal have 
a representation as a finite intersection of s-right primary ideals, which is 
analogous to that in [4]. In case the maximum condition is satisfied for 
ideals, the first half of our condition can be excluded (§ 4).

1. Definitions. Let \( R \) be a (non-commutative) ring. The term "ideal" 
in \( R \) will always mean "two-sided ideal".

Definition 1. \( A \) and \( B \) are ideals in \( R \), the ideal consisting of all 
elements \( x \) of \( R \) such that \( xRB \subseteq A \) is called the right ideal quotient of \( A \) by 
\( B \) and is denoted by \( AB^{-1} \). Similary, \( B^{-1}A \) consists of all \( x \) in \( R \) such that 
\( BRx \subseteq A \).

The following properties of quotients are verified:

1. \( (AB^{-1})C^{-1} = A(CB)^{-1} \),
2. \( (\bigcap A_i)B^{-1} = \bigcap A_i B^{-1} \),
3. \( A(\sum B_a)^{-1} = \bigcap A B_a^{-1} \), where \( A, B, C, A, \) and \( B_a \) are ideals in \( R \).

Definition 2. An element \( a \) is right non-prime to an ideal \( A \) if there 
exists an element \( b \) not in \( A \) such that \( bRa \subseteq A \). An ideal \( B \) is right non- 
prime to \( A \) if \( AB^{-1} \supset A \).

For positive integers \( n \) we define inductively \( AB^{-n} = (AB^{-1})B^{-1} \). If 
\( AB^{-k} = AB^{-k+1} \) for some positive integer \( k \) then we say that \( AB^{-k} \) is the 
right limit ideal of \( A \) by \( B \). The left limit ideal \( B^{-k}A \) can be defined in 
the same way. An ideal \( P \) in \( R \) is prime if \( AB \subseteq P \) implies that either \( A \subseteq P \) 
or \( B \subseteq P \), where \( A \) and \( B \) are ideals in \( R \). It has been shown by McCoy 
[2] that an ideal \( P \) is prime if and only if \( aRb \subseteq P(a, b \in R) \) implies that

\(^1\) "s-right primary" means "strongly right primary".
either \(a\) or \(b\) belongs to \(P\). The \textit{radical} of an ideal \(A\) is understood in the sense of McCoy [2] and denoted by \(r(A)\). It has been shown by McCoy [2] that \(r(A)\) is the intersection of all minimal prime divisors of \(A\).

**Definition 3.** An ideal \(Q\) is said to be \textit{right primary} if \(ab \subseteq Q\) and \(a \notin Q\) imply \(b \in r(Q)\), and a right primary ideal \(Q\) is defined to be \textit{s-right primary} if \(r(Q)\) is nilpotent modulo \(Q\).

One will easily see that an ideal \(Q\) is \(s\)-right primary if and only if it is \(s\)-right primary in Tominaga's sense, and so the radical of an \(s\)-right primary ideal is prime by Theorem 1 of [4].

**Definition 4.** If a prime ideal \(P\) is the radical of an \(s\)-right primary ideal \(Q\), we say that \(Q\) \textit{belongs to} \(P\) and also that \(Q\) is \(P\)-\textit{s-right primary}. A prime ideal \(P\) is called a \textit{prime ideal associated with} an ideal \(A\) if there exists an \(s\)-right primary ideal \(Q\) belonging to \(P\) such that \(Q = B^{-1}A\) for some ideal \(B\) not contained in \(A\).

2. **Uniqueness theorem for \(s\)-right primary representations.**

A representation \(A = Q_1 \cap Q_2 \cap \cdots \cap Q_n\) of an ideal \(A\) as the intersection of \(s\)-right primary ideals \(Q_1, Q_2, \cdots, Q_n\) will be called \textit{irredundant} if no one of the \(Q_i\) contains the intersection of the remaining ones.

**Theorem 1.** Let \(A = Q_1 \cap Q_2 \cap \cdots \cap Q_n\) be an irredundant representation of \(A \subseteq R\), where \(Q_i\) is \(P_i\)-\(s\)-right primary \((1 \leq i \leq n)\). Then an element \(x\) is right non-prime to \(A\) if and only if \(x \in P_j\) for some \(j\), namely, \(x \in P_1 \cup P_2 \cup \cdots \cup P_n\).

**Proof.** If \(x\) is right non-prime to \(A\) then \(bRx \subseteq A\) for some \(b\) not in \(A\). But this implies \(bRx \subseteq Q_i\) \((1 \leq i \leq n)\), while \(b \notin Q_j\) for some \(j\). Since \(Q_j\) is \(P_j\)-\(s\)-right primary, we obtain \(x \notin P_j\). Conversely, suppose that \(x\) is in \(P_j\). Since the representation \(A = Q_1 \cap Q_2 \cap \cdots \cap Q_n\) is irredundant, we can choose an element \(b\) which is contained in \(Q_1 \cap \cdots \cap Q_{j-1}\) but not in \(Q_j\). Noting that \((RP_j)^k \subseteq Q_i\) for some \(k\), we have then \(b(RP_j)^k \subseteq A\). Accordingly, there exists the least positive integer \(k_i\) such that \(b(RP_j)^{k_i} \subseteq A\). If \(k_i = 1\) then \(bRP_j \subseteq A\). Hence \(bRx \subseteq A\). Thus, \(x\) is right non-prime to \(A\). If \(k_i > 1\) then the product \(b(RP_j)^{k_i-1}\) contains an element \(b_i\) not in \(A\). Since \(b_iRx \subseteq A\), \(x\) is right non-prime to \(A\).

**Lemma 1.** If \(Q_1, Q_2, \cdots, Q_n\) are \(P\)-\textit{s-right primary ideals} then \(Q = Q_1 \cap Q_2 \cap \cdots \cap Q_n\) is also a \(P\)-\textit{s-right primary ideal}.

**Proof.** Let \(k_i\) be the nilpotency index of \(P\) modulo \(Q_i\) \((1 \leq i \leq n)\). Then, \(P^{k_1 + \cdots + k_n} \subseteq Q\). If \(P_i\) is any prime divisor of \(Q_i\), we have \(P^{k_1 + \cdots + k_n} \subseteq P_i\), whence it follows \(P \subseteq P_i\). Hence, \(P\) is a unique minimal prime divisor of \(Q\) and therefore \(P = r(Q)\). Moreover, if \(aRb \subseteq Q\) and \(a \notin Q\) then \(aRb \subseteq Q_i\) \((1 \leq i \leq n)\), while \(a \notin Q_j\) for some \(j\). Since \(Q_j\) is \(P\)-\textit{s-right primary}, this implies that \(b \in P = r(Q_j)\). Hence, \(Q\) is \(P\)-\textit{s-right primary}.
By the same argument as in Theorem 14 of [3] we have

Lemma 2. If \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) is an irredundant representation of \( A \), where \( Q_i \) is \( P_i \)-s-right-primary \((1 \leq i \leq n)\) and \( P_j \neq P_k \) for some \( j \neq k \), then \( A \) is s-right primary.

Definition 5. An irredundant representation \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) will be called a short representation if none of the intersections of two or more of the ideals \( Q_1, Q_2, \ldots, Q_n \) are s-right primary.

In view of Lemmas 1 and 2, an irredundant representation \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) is a short representation if and only if any two of the radicals of \( Q_1, Q_2, \ldots, Q_n \) are distinct.

Let \( M \) be a non-empty \( m \)-system in \( R \). For any ideal \( A \) in \( R \) the right upper and lower isolated \( M \)-components of \( A \) (in the sense of [3]) will be denoted by \( U(A, M) \) and \( L(A, M) \), respectively. If \( P \) is a prime ideal \((P \neq R)\) and \( M = C(P) \) is its complement in \( R \) then \( U(A, M) \) will be denoted by \( U(A, P) \).

Theorem 2. Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) be an irredundant representation of \( A \), where \( Q_i \) is \( P_i \)-s-right primary \((1 \leq i \leq n)\). If \( M(\subseteq R) \) is a non-empty \( m \)-system which does not meet \( P_1, \ldots, P_r, \) but meets \( P_{r+1}, \ldots, P_s, \) then \( U(A, M) = L(A, M) = Q_1 \cap Q_2 \cap \cdots \cap Q_r \). If \( M \) meets every \( P_i \), then \( U(A, M) = L(A, M) = R \).

Proof. By the same argument as in Theorem 15 of [3], we can easily see that if \( M \) does not meet \( P_1, \ldots, P_r \), but meets \( P_{r+1}, \ldots, P_s \), then \( U(A, M) = Q_1 \cap Q_2 \cap \cdots \cap Q_r \), and that if \( M \) meets every \( P_i \) then \( U(A, M) = R \).

We assume first that \( M \) does not meet \( P_i \), \( \ldots \), \( P_r \) but meets \( P_{r+1}, \ldots, P_s \). Let \( b \) be an element of \( L(A, M) \). Then we have \( bRm \subseteq A \) for some \( m \in M \) and thus \( bRm \subseteq Q_i(1 \leq i \leq r) \). However, \( m \) is not in any \( P_i(1 \leq i \leq r) \). Hence \( b \in Q_i(1 \leq i \leq r) \) and thus \( L(A, M) \subseteq Q_1 \cap Q_2 \cap \cdots \cap Q_r \). We shall prove now the converse inclusion. If \( r = n \) then this is trivial by \( A \subseteq L(A, M) \). In case \( r < n \), since \( M \) meets \( P_j \) for \( j > r \), it follows that \( M \) meets \( Q_i \) for \( j > r \). Hence there exist \( m_1, m_2, \ldots, m_{n-r} \) such that \( m_i \in Q_{r+i} \cap M(1 \leq i \leq n-r) \). Now, since every \( m_i \) is in \( M \), there exist \( x_1, x_2, \ldots, x_{n-r} \) such that \( m = m_1 x_1 m_2 x_2 \cdots x_{n-r} m_{n-r} \) is contained in \( M \). Since it is clear that \( m \in Q_{r+1} \cap Q_{r+2} \cap \cdots \cap Q_n \), \( qRm \subseteq A \) for every element \( q \in Q_1 \cap Q_2 \cap \cdots \cap Q_r \). Thus \( q \) is in \( L(A, M) \).

If \( M \) meets every \( P_i \), then the last part of the above proof shows that there is an element \( m \in M \) such that \( m \in Q_1 \cap Q_2 \cap \cdots \cap Q_n = A \). Hence \( Rm \subseteq A \) for every \( R \in R \), that is, \( R = L(A, M) \).

Theorem 3. Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) be an irredundant representation of \( A \), where \( Q_i \) is \( P_i \)-s-right primary \((1 \leq i \leq n)\). Then the minimal prime divisors of \( A \) are exactly those primes which are minimal in the set \( \{ P_1, P_2, \ldots, P_n \} \).
Proof. This is immediate.

Theorem 4. Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) be a short representation of \( A \subset R \), where \( Q_i \) is \( P_i \)-s-right primary \((1 \leq i \leq n)\). A prime divisor \( P(\neq R) \) of \( A \) is one of \( P_i \) if and only if every element of \( P \) is right non-prime to \( U(A, P) \). The ring \( R \) is itself one of the \( P_i \) if and only if every element of \( R \) is right non-prime to \( A \).

Proof. Let \( P(\neq R) \) be a prime divisor of \( A \). If \( P \) coincides with one of \( P_i \), then by Theorem 2 \( U(A, P) = Q_1 \cap Q_2 \cap \cdots \cap Q_i \) is a short representation of \( U(A, P) \), where \( P_{i_1}, P_{i_2}, \cdots, P_{i_r} \) are those primes among \( \{ P_i \} \) which are contained in \( P \) (and so \( P \) is maximal among them). Hence, by Theorem 1, every element of \( P \) is right non-prime to \( U(A, P) \). Conversely, assume that every element of \( P \) is right non-prime to \( U(A, P) \). By Theorem 3, \( P \) contains at least one of \( P_i \). Suppose that \( P \) contains \( P_1, \cdots, P_i \) but does not contain \( P_{i+1}, \cdots, P_n \). Then, again by Theorem 2, \( U(A, P) = Q_1 \cap Q_2 \cap \cdots \cap Q_i \) is a short representation of \( U(A, P) \). Hence, \( P \subset P_1 \cup P_2 \cup \cdots \cup P_i \), by Theorem 1, and then by Theorem 5 of [1] there exists some \( i \) such that \( P \subset P_i \), namely, \( P = P_i \). The latter assertion is also an easy consequence of Theorem 5 of [1] and Theorem 1.

As an immediate consequence of Theorem 4, we obtain the following:

Theorem 5. Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n = Q'_1 \cap Q'_2 \cap \cdots \cap Q'_m \) be two short representations of \( A \), where \( Q_i \) is \( P_i \)-s-right primary and \( Q'_i \) is \( P'_i \)-s-right primary. Then, \( m = n \) and it is possible to number the components in such a way that \( P_i = P'_i \) \((1 \leq i \leq m = n)\).

Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) be a short representation of \( A \), where \( Q_i \) is \( P_i \)-s-right primary \((1 \leq i \leq n)\). These uniquely determined prime ideals \( P_1, P_2, \cdots, P_n \) will be called the prime ideals belonging to \( A \) (cf. Theorem 5). A subset \( \{ P_{i_1}, P_{i_2}, \cdots, P_{i_r} \} \) of these prime ideals is called an isolated set of prime ideals belonging to \( A \) if every \( P_j \) contained in one of the primes \( P_{i_1}, P_{i_2}, \cdots, P_{i_r} \) is necessarily a member of the subset.

Now, by Theorem 2, one will readily obtain the following:

Theorem 6. Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) be a short representation of \( A \), where \( Q_i \) is \( P_i \)-s-right primary \((1 \leq i \leq n)\). If \( \{ P_{i_1}, P_{i_2}, \cdots, P_{i_r} \} \) is an isolated set of prime ideals belonging to \( A \) then \( Q_{i_1} \cap Q_{i_2} \cap \cdots \cap Q_{i_r} \) depends only on \( \{ P_{i_1}, P_{i_2}, \cdots, P_{i_r} \} \) and not on the particular short representation considered.

3. A necessary and sufficient condition that every ideal be represented as a finite intersection of s-right primary ideals.

Theorem 7. Let \( A = Q_1 \cap Q_2 \cap \cdots \cap Q_n \) be a short representation of \( A \subset R \), where \( Q_i \) is \( P_i \)-s-right primary \((1 \leq i \leq n)\). If \( P \) is a minimal prime
divisor of $A$ then $P$ is right non-prime to $A$.

Proof. By Theorem 3, we can assume that $P$ is contained in $P_1, \ldots, P_r (r \geq 1)$ but not contained in $P_{r+1}, \ldots, P_n$. Then $R = Q_1P^{-k} = Q_1P^{-k+1}$ for a sufficiently large positive integer $k (1 \leq i \leq r)$. On the other hand, if $r + 1 \leq j \leq n$ then $Q_j = Q_jP^{-k} = Q_jP^{-k+1}$ for every positive integer $k$. Thus $AP^{-k} = AP^{-k+1} = Q_{r+1} \cap Q_{r+2} \cap \cdots \cap Q_n$. Since $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ is a short representation, we have $AP^{-k} \supset A$, and therefore $AP^{-k} \supset A$.

Lemma 3. If $Q$ is a $P$-right primary ideal then $B^{-1}Q$ is $P$-right primary for any ideal $B \not\supseteq Q$.

Proof. Since $BR(B^{-1}Q) \subseteq Q$ and $B \not\supseteq Q$, we have $Q \subseteq B^{-1}Q \subseteq P$, and thus $r(B^{-1}Q) = P$. Suppose that $aRb \subseteq B^{-1}Q$ and $b \not\in P$. Then we have $BRA \subseteq Q$. Hence, by the definition of $s$-right primary, $BRA \subseteq Q$, that is, $a \in B^{-1}Q$.

By the same arguments as in Theorems 4 and 6 of [4], we have the following two theorems.

Theorem 8. Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a short representation of $A$. Then, for any ideal $B$ there exists the right limit ideal of $A$ by $B$, and the number of ideals which are obtained starting from $A$ by repeating successively the procedure to make right limit ideals is finite and is uniquely determined by $A$.

Theorem 9. Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a short representation of $A \subseteq R$, where $Q_i$ is $P_i$-right primary $(1 \leq i \leq n)$. Then, a prime divisor $P$ of $A$ is a prime ideal associated with $A$ if and only if $P$ coincides with one of $P_i$, and every primary component $Q_i (1 \leq i \leq n)$ has the following property: $B^{-1}A$ is not $P_i$-right primary for any ideal $B$ such that $B \supseteq Q_i$, and $B \not\supseteq A$.

Corollary 1. Let $A = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ be a short representation of $A \subseteq R$. If $P$ is a minimal prime divisor of $A$ then $P$ is a prime ideal associated with $A$.

Now, we can summarize the above-mentioned results as follows:

Theorem 10. In order that every ideal in $R$ be represented as the intersection of a finite number of $s$-right primary ideals, the following conditions are necessary:

(A) For any ideals $A$, $B$ in $R$ there exists the right limit ideal of $A$ by $B$ and there exist a finite number $n(A)$ of ideals which are obtained starting from $A$ by repeating successively the procedure to make right limit ideals, where the number $n(A)$ is uniquely determined by $A$.

(B) Every ideal $A \subseteq R$ has a minimal prime divisor which is right non-prime to $A$.

(C) Every minimal prime divisor of an arbitrary not $s$-right primary
ideal $A$ is a prime ideal associated with $A$.

(D) If $P$ is an arbitrary prime ideal associated with an ideal $A$ then there exists an $s$-right primary ideal $Q \supseteq A$ belonging to $P$ such that $B^{-1}A$ is not $P$-s-right primary for any subideal $B$ of $Q$ not contained in $A$.

Next, we shall show that these conditions are sufficient, too.

**Lemma 4.** Assume the conditions (A) and (B) in Theorem 10. If $A$ is an ideal of $R$ then $r(A)$ is nilpotent modulo $A$.

**Proof.** Let $P$ be a minimal prime divisor of $A \subseteq R$ which is right non-prime to $A$. Then $A \subseteq AP^{-1} \subseteq Ar(A)^{-1}$. If $Ar(A)^{-1}$ is not $R$ itself then we have $Ar(A)^{-1} \subseteq Ar(A)^{-1}r(Ar(A)^{-1})^{-1} \subseteq Ar(A)^{-1}$. Continuing in this way, we obtain the right limit ideal $Ar(A)^{-k}$ of $A$ by $r(A)$. We have then $Ar(A)^{-k} = R$, whence it follows $r(A)^{2k+1} \subseteq A$.

By the same argument as in Lemma 4 of [4], we have the following:

**Lemma 5.** Assume the conditions (A), (B) and (C) in Theorem 10. Then the number of prime ideals associated with an ideal which is not $s$-right primary is finite.

We assume here the conditions (A), (B), (C) and (D) in Theorem 10. Let $P_1, P_2, \ldots, P_n$ be all the prime ideals associated with an ideal $A$ which is not $s$-right primary. and let $Q_1, Q_2, \ldots, Q_n$ be $s$-right primary divisors of $A$ belonging to $P_1, P_2, \ldots, P_n$ with the property cited in (D), respectively (Lemma 5). We set $B = Q_1 \cap Q_2 \cap \cdots \cap Q_n$. By the condition (C), every minimal prime divisor of $A$ is a prime ideal associated with $A$, and so $B \subseteq r(A)$. Since $r(A)$ is nilpotent modulo $A$ by Lemma 4, we obtain $B^{-1}A \supseteq A$. We suppose now that $B \supseteq A$. If $B^{-1}A$ is not $s$-right primary then by the condition (C) we have an $s$-right primary ideal $C \subseteq B^{-1}A$ for some $C \notin A$. So we set $C = BRC_\circ$. If $B^{-1}A$ is $s$-right primary, we set $C = B$. Thus, in either case, we have an $s$-right primary ideal $Q = C^{-1}A$, where $C \subseteq A$ and $C \subseteq B$. Since $r(Q)$ is a prime ideal associated with $A$, $r(Q) = P_i$ for some $i$. On the other hand, since $C \subseteq B \subseteq Q$, the ideal $Q = C^{-1}A$ is not $P_i$-s-right primary by the condition (D). This contradiction means $A = B$. Hence, we have the following theorem.

**Theorem 11.** In order that every ideal in $R$ be represented as the intersection of a finite number of $s$-right primary ideals, it is necessary and sufficient that the conditions (A), (B), (C) and (D) be satisfied.

4. **Rings with maximum condition for ideals.**

Throughout the present section, $R$ be a ring with maximum condition for ideals. Then, needless to say, for any ideals $A, B$ of $R$ there exists the right limit ideal of $A$ by $B$.

**Lemma 6.** Every ideal $A \subseteq R$ has a minimal prime divisor which is right non-prime to $A$. 

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Proof. One may assume that $A$ is not prime. By Theorem 10 of [3], we have $P_1 P_2 R \cdots R P_s \subseteq A$, where $P_1, \ldots, P_s$ are minimal prime divisors of $A$ and $s > 1$. Hence we can assume that $P_1 P_2 R \cdots R P_s \subseteq A$ and $P_1 R P_3 R \cdots R P_{s-1} \subseteq A$. If $b$ is an arbitrary element of $P_1 R P_3 R \cdots R P_{s-1}$, not contained in $A$ then $b R P_s \subseteq A$, and so $P_s$ is right non-prime to $A$.

From the proof of Lemma 6, the following will be obvious.

Corollary 1. If $A$ is an ideal of $R$ then $r(A)$ is nilpotent modulo $A$. In particular, every primary ideal of $R$ is s-right primary.

Lemma 7. Assume the condition (C). If an ideal $A$ is not right primary then the number of prime ideals associated with $A$ is finite.

Proof. Let $\{ P_i \}$ be the set of all prime ideals associated with $A$, and let $Q_i = P_i^{-1} A$ be a $P_i$-right primary ideal. The set $\{ P_i \}$ is not empty by the condition (C). Let $\{ P_1, P_2, \ldots, P_s \}$ be a subset in $\{ P_i \}$ such that $P_i \not\subseteq P_j$ for every $i > j$. We define now the ideals $B'_1, B'_2, \ldots, B'_k$ in the following way: $B'_1$ is the right limit ideal of $A$ by $P_1$, and $B'_i$ is the right limit ideal of $B'_{i-1}$ by $P_i (i = 2, \ldots, k)$. Then, by the analogous argument as in Lemma 4 of [4], we have an ascending chain $A \subseteq B'_1 \subseteq B'_2 \subseteq \cdots \subseteq B'_k$. From this fact, the lemma will be easily seen.

Now, by the validity of Lemmas 6, 7 and Corollary 1 to Lemma 6, the proof of the following theorem proceeds just like that of Theorem 11 did.

Theorem 12. Let $R$ be a ring with maximum condition for ideals. In order that every ideal in $R$ can be represented as the intersection of a finite number of right primary ideals, it is necessary and sufficient that the conditions (C) and (D) be satisfied.

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