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ALGEBRAIC EXTENSIONS OF SIMPLE RINGS I

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In this paper we study the local finiteness of algebraic extensions of simple rings. Our study starts with the preliminary section § 1, which contains some matrix calculations of independent interest. In § 2, we consider simple algebras of finite rank, and show under what conditions they are locally finite, of bounded degree over simple subrings. In § 3, we study the equality of left and right dimensionalities for some types of simple ring extensions. In § 4, we consider QG-1 extensions of simple rings which are generalized Galois extensions, and in § 5, we study the local finiteness of QG-1 division ring extensions of infinite rank. For QG-1 extensions of simple rings, the situations will be studied in a continuation of the present title, II.

Throughout the present paper, $A=\sum_1^n De_{ij}$ will represent a simple ring where $E=\{e_{ij}\}$ is a system of matrix units and D is the centralizer of E in A which is a division ring, and B a simple subring of A containing the identity 1 of A. For any subset S of A, we denote by $V_A(S)$ the centralizer of S in A. Specially we use the following convention: V means $V_A(B)$. Moreover we write $C=V_A(A)$, $Z=V_B(B)$, $C_0=V_V(V)$ the centers of A, B, and V respectively. If, for every finite subset F of A, $[B[F]:B]_L<\infty$ (resp. $[B[F]:B]_R<\infty$) then we say that A is left (resp. right) locally finite over B. Given rings R, S, we shall understand by an R-S-module a two-sided module which is treated as a left R- and right S-module. For any simple subring T of A, we denote by $[T|T]$ the uniquely determined number of irreducible direct summands of T-module T. If $T\supset E$ then one can see that $[T|T]=[A|A]$. A subring T of A is said to be regular if T and $V_A(T)$ are both simple. By $\mathfrak{R}$ we denote the set of all regular subrings of A containing B, and we shall use the following notations:

$\mathfrak{R}^0=\{B'\in\mathfrak{R}\,;\,[B'|B']=[A|A]\}$,

$\mathfrak{R}_{l.f}=\{B'\in\mathfrak{R}\,;\,[B':B]_L<\infty\}(\mathfrak{R}_{r.f}=\{B'\in\mathfrak{R}\,;\,[B':B]_R<\infty\})$,

$\mathfrak{R}/S=\{B'\in\mathfrak{R}\,;\,B'\supset S\}$, where S is a subset of A, and we set $\mathfrak{R}^0_{l.f}=\mathfrak{R}^0\cap\mathfrak{R}_{l.f}$, $\mathfrak{R}^0/S=\mathfrak{R}^0\cap\mathfrak{R}/S$, $\mathfrak{R}_{l.f}/S=\mathfrak{R}_{l.f}\cap\mathfrak{R}/S$, $\mathfrak{R}^0_{l.f}/S=\mathfrak{R}^0_{l.f}\cap\mathfrak{R}/S$. As to other notations and terminologies used in this paper, we follow [3—11].

1. Preliminaries

We consider the following

$$u(E, d) = de_{21} + \sum_3^n e_{ii-1}$$

with non-zero $d \in D$, which plays an important role in our subsequent consideration. First we shall prove the following

Lemma 1. *Let $n>1$, and T a left Artinian unital subring of A. If T contains $u(E, d)$ and $a=\sum x_{ij}e_{ij}$ $(x_{ij}\in D)$ with $x_{1n}=1$ and $x_{in}=0$ for every $i>1$, then T contains E, d, and x_{ij}'s.*

Proof. Since A is T-A-irreducible by [7, Lemma 8], T is in $\mathfrak{R}$ (cf. [11, p. 69]). We set $u=u(E, d)$. Then $u^{k-1}au^{n-1}=d^2e_{k1}$ is a non-zero element of $T\cap e_{kk}A$ $(k=1, \cdots, n)$, and so by [7, Lemma 5] T contains e_{11}, e_{22}, $\cdots$, e_{nn}. It follows therefore $e_{1n}=e_{11}ae_{nn}\in T$ and $d=(u+e_{1n})^n \in T$, so that $e_{21}=d^{-2}(d^2e_{21})\in T$. It follows then $\sum_2^n e_{ii-1}=(1-d)e_{21}+u\in T$, and therefore our assertion is a direct consequence of [7, Lemma 6].

The next contains evidently [1, Satz].

Corollary 1. *Let $\Phi(\subset C)$ be a perfect field. If A/Φ is 3-algebraic then A/Φ is locally finite, In case $n>1$, if A/Φ is 2-algebraic then A/Φ is locally finite.*

Proof. Firstly, assume that A is a division ring and 3-algebraic over Φ. If a_1, a_2, a_3 are in A then $\Phi[a_1, a_2, a_3]=\Phi[a'_1, a'_2]$ with some a'_1, $a'_2\in A$ ([7, Th. 1]). Hence, an easy induction will prove that A/Φ is locally finite. Next, assume that $n>1$ and A/Φ is 2-algebraic. Let d_1, d_2, d_3 be non-zero elements of D. If $a_1=\sum x_{ij}e_{ij}$ where $x_{11}=d_1$, $x_{21}=d_2$, $x_{1n}=1$ and each of other x_{ij}'s is 0, and $a_2=u(E, d_3)$, then Φ $[a_1, a_2]$ is finite over Φ and contains $\Phi[d_1, d_2, d_3, E]$ (Lemma 1), which means that D/Φ is 3-algebraic. Hence, A/Φ is locally finite by the first assertion proved above.

Proposition 1. *Let A be left algebraic (resp. left algebraic and of bounded degree) over B. If $B \not\subset C$ then there exists some $B'=\sum_1^n D'e'_{ij}\in \mathfrak{R}_{l.f}^0$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic (resp. left algebraic and of bounded degree).*

Proof. It suffices to prove the proposition for the case $n>1$. By [7, Lemma 7], there exists an element $r\in A^{\cdot}$ such that $B\tilde{r}$ contains an element $a=\sum x_{ij}e_{ij}(x_{ij}\in D)$ with $x_{1n}=1$ and $x_{in}=0$ for every $i>1$. Given an arbitrary non-zero element $d\in D$, we set $u=u(E, d)$. Then, $(B\tilde{r})[u]$ is left finite over $B\tilde{r}$ and contains $\{E, d\}$ (Lemma 1). If we set $B^*=(B\tilde{r})[E]=\sum D^*e_{ij}$ with the division ring $D^*=V_{B^*}(E)$, then D/D^* is left algebraic. Hence, $B^* \tilde{r}^{-1}$ can be taken as B' requested.

Proposition 2. *Let* $[B:Z]<\infty$ (*or* $[A:C]<\infty$). *If* $[A:B]_L<\infty$ *then* $[A:B]_R<\infty$.

Proof. By [9, Lemma], we obtain $[A:C]<\infty: A=\sum_1^s a_i C$. Since $C\cdot B=\sum_1^t Bc_j$ $(c_j \in C)$, it follows $\sum_{i,j} a_i c_j B=\sum_i a_i C\cdot B=A$, which means $[A;B]_R<\infty$.

Corollary 2. *Let* $[B:Z]<\infty$. *If* A/B *is left locally finite then it is locally finite.*

2. Algebraic extensions of a simple algebra of finite rank

The present section is devoted exclusively to the treaty of algebraic extensions of a simple algebra of finite rank.

Proposition 3. *Let* A *be left algebraic over* B.

(a) *In order that* $[A:C]<\infty$, *it is necessary and sufficient that* $[V:C]<\infty$ *and* $[B:Z]<\infty$.

(b) *If* $[A:C]<\infty$ *then* A/B *is locally finite.*

Proof. (a) By the validity of [10, Lemma], it suffices to prove the sufficiency. In case B is contained in C, our assertion is trivial. If B is not contained in C then there exists some $B'=\sum_1^{n'} D'e'_{ij} \in \mathfrak{R}_{l,f}^0$ such that $V_A(\{e'_{ij}\})$ is left algebraic over D' (Prop. 1). Since $[B':V_{B'}(B')]<\infty$ again by [10, Lemma], we may assume from the beginning that A is a division ring. Then, noting that $B\cdot C=B\otimes_Z Z\cdot C$, there holds $[B\cdot C:C]=[B\cdot C:Z\cdot C]\cdot[Z\cdot C:C]\leqslant[B:Z]\cdot[V:C]<\infty$. Hence, $B\cdot C=V_A^2(B\cdot C)=V_A(V)$, and so $[A:C]=[A:B\cdot C]\cdot[B\cdot C:C]=[V:C]\cdot[B\cdot C:C]<\infty$.

(b) In case B is contained in C, our assertion is clear by [2, Prop. 10.12.3] or by a direct computation. In what follows, we shall assume that $B\not\subset C$. Then, by [7, Th. 1], there exists an element $a\in A$ such that $A=(B\cdot C)[a]$. Now, let F be an arbitrary finite subset of A. Then, C contains a finite subset F' such that $B[F]\subset B[F',a]=B[a]\cdot B[F']$. Since we can easily see that $[B[F']:B]_L<\infty$, our assertion is a consequence of (a) and Cor. 2.

The next theorem contains obviously Prop. 3 (b).

Theorem 1. *Let* $[B:Z]<\infty$, *and let* A *be left algebraic over* B. *In order that* A/B *be locally finite, it is necessary and sufficient that* $A/B\cdot C$ *be left locally finite.*

Proof. It suffices to prove the sufficiency. At first we shall prove our assertion for the case that B is a regular subring. Since $B\cdot C=B\otimes_Z C\cdot Z$, $Z\cdot C$ is a subfield of C_0 and $B\cdot C$ is a simple ring with $[B\cdot C:Z\cdot C]=[B:Z]<\infty$. Now, let F be an arbitrary finite subset of A. Then, $A'=(B\cdot C)[E, F]$ is a simple ring with $[A':B\cdot C]_L<\infty$. Recalling here that $[B\cdot C:Z\cdot C]<\infty$, we have then $[A':V_{A'}(A')]<\infty$ by [10, Lemma]. Hence, A'/B is locally finite by Prop. 3 (b), which means obviously that A/B is locally finite. Next, we shall proceed into the general case. By the first step, we may restrict our attention to the case that $B\not\subset C$. Then, by Props. 1 and 2 (a), there exists some $B'=\sum_1^n D'e'_{ij}\in\mathfrak{R}_{l.f}=\mathfrak{R}^0_{r.f}$ such that $V_A(\{e'_{ij}\})$ is left algebraic over D'. One may remark here $[D':V_{D'}(D')]\leqslant[B':V_{B'}(B')]<\infty$ ([10, Lemma]). Since $B'\cdot C=\sum_1^n(D'\cdot C)e'_{ij}$ is left finite over $B\cdot C$, $A/B'\cdot C$ is left locally finite, or what is the same, $V_A(\{e'_{ij}\})/D'\cdot C$ is left locally finite. Hence, by the first step, $V_A(\{e'_{ij}\})/D'$ is locally finite, or equivalently A/B' is locally finite. Consequently, it follows that A/B is locally finite.

Theorem 2. *Let A be left algebraic over a simple ring B. Then, the following conditions are equivalent :*

(1) $[B:Z]<\infty$ *and A/B is of bounded degree.*

(2) $[A:C]<\infty$ *and $B\cdot C/B$ is of bounded degree.*

(3) $[A:C]<\infty$ *and $Z\cdot C/Z$ is of bounded degree.*

Proof. Evidently, $B\cdot C=B\otimes_Z Z\cdot C$ and (2)⇒(3).

(3)⇒(1). We set $A=\sum_1^s Ca_i$, where $a_1=1$. There exists a positive integer k such that every subring of the form $B[c]$ $(c\in C)$ possesses a linearly independent B-basis consisting of at most k elements of C. Accordingly, if $c_1,\cdots,c_t$ are in C then $B[c_1,\cdots,c_t]$ possesses a linearly independent B-basis consisting of at most k^t elements of C. In case $B\subset C$, the last yields $[B[x]:B]\leqslant s.\,k^{s^2+s}$ for every $x\in A$. On the other hand, if $B\not\subset C$ then $A=(B\cdot C)[a]$ with some a ([7, Th. 1]). There exists then a finite subset F of C such that $\{a_1,\cdots,a_s\}\subset B[F,a]$. If $x=\sum_1^s c'_ia_i$ is an element of A and $F'=\{c'_1,\cdots,c'_s\}$, it is obvious that $B[x]\subset B[F,F',a]$. Hence, $[B[x]:B]_L\leqslant[B[F,F',a]:B]_L=[B[a]\cdot B[F,F']:B]\leqslant[B[a]:B]\cdot k^{\sharp F+s}$. One may remark here $[B:Z]\leqslant[A:C]<\infty$ ([10, Lemma] or Prop. 3).

(1)⇒(2). If $B\subset C$ then A/C is of bounded degree, and then $[A:C]<\infty$ by [4, Lemma 4]. Next, assume that $B\not\subset C$. By Prop. 1, there exists some $B'=\sum_1^n D'e'_{ij}\in\mathfrak{R}^0_{l.f}$ such that $V_A(\{e'_{ij}\})/D'$ is left algebraic and of bounded degree. Since $[D':V_{B'}(B')]\leqslant[B':V_{B'}(B')]<\infty$ by [10, Lemma], $V_A(\{e'_{ij}\})$ is left algebraic and of bounded degree over the field $V_{B'}(B')\cdot C$. Hence, again by [4, Lemma 4], $[V_A(\{e'_{ij}\}):C]<\infty$, namely, $[A:C]<\infty$.

3. Left and right dimensionalities

Proposition 4. *Let $[B:Z]<\infty$ (or $[A:C]<\infty$). If B is a regular subring then $[A:B]_L=[A:B]_R$, provided we do not distinguish between two infinite dimensions.*

Proof. Assume that $[A:B]_L<\infty$. Since $B\cdot V=B\otimes_Z V$ is a simple intermediate ring of A/C and $[A:C]<\infty$ by [10, Lemma], we obtain $B\cdot V=V_A^2(B\cdot V)=V_A(C_0)$ and $[B\cdot V:B]=[V:Z]$. Hence, $[A:B]_L=[A:B\cdot V]_L\cdot[B\cdot V:B]=[C_0:C]\cdot[B\cdot V:B]=[A:B\cdot V]_R\cdot[B\cdot V:B]=[A:B]_R$, which proves our assertion.

As an application of Th. 2, we shall prove the following that contains [6, Th. 5.2].

Theorem 3. *Let $[B:Z]<\infty$, and let A be left algebraic and of bounded degree over B. If $Z\cdot C$ is a separable field extension of Z then $[A:B]_L=[A:B]_R<\infty$.*

Proof. By Th. 2, there holds $[A:B]_L\leqslant[A:C]\cdot[Z\cdot C:Z]<\infty$. Since the simple ring $V_A(Z)=V_A(Z\cdot C)$ is $B\otimes_Z V$ by Wedderburn' theorem, V is a simple ring. Now, our assertion is a consequence of Prop. 4.

Throughout the rest of this note, $\mathfrak{A}$ and $\mathfrak{G}$ will denote the absolute endomorphism ring Hom (A, A) of A and the group of all B-ring automorphisms of A, respectively. We now consider some results which can be proved for Galois extensions A/B.

Proposition 5. *Let $\mathfrak{G}A_R$ and $\mathfrak{G}A_L$ be dense in $V(B_L)$ and $V_{\mathfrak{A}}(B_R)$, respectively. Let B' be an intermediate ring of A/B left (or right) finite over B such that A is B'-A-irreducible and A-B'-irreducible. In order that $[B':B]_L=[B':B]_R$, it is necessary and sufficient that $[V:V_A(B')]_R=[V:V_A(B')]_L$.*

Proof. If $\sigma|B'=\tau\tilde{v}|B'$ for some $v\in V^{\cdot}$ ($\sigma,\tau\in\mathfrak{G}$), we write $\sigma|B'\sim\tau|B'$. Evidently, the relation $\sim$ is an equivalence relation in $\mathfrak{G}|B'$, and the number of the equivalence classes w. r. t. $\sim$ denoted as $(\mathfrak{G}|B':\tilde{V})$ coincides with the number of B'_R-A_R-homogeneous components of $(\mathfrak{G}|B')V_LA_R$ as well as with that of B'_L-A_L-homogeneous components of $(\mathfrak{G}|B')V_RA_L$. Hence, by [8, Lemmas 1.3, 1.4, 1.5], we obtain $[B':B]_L=(\mathfrak{G}|B':\tilde{V})\cdot[V:V_A(B')]_R$ and $[B':B]_R=(\mathfrak{G}|B':\tilde{V})\cdot[V:V_A(B')]_L$. Comparing those above, we readily see our conclusion.

Combining Prop. 5 with [9, Cor. 2 (b)], we obtain the next:

Corollary 3. *Let $J(\mathfrak{G}, A)=B$, and let A be $B\cdot V$-A-irreducible and*

A-$B\cdot V$-irreducible. Let B' be an intermediate ring of A/B left (or right) finite over B such that A is B'-A-irreducible and A-B'-irreducible. In order that $[B':B]_L=[B':B]_R$, it is necessary and sufficient that $[V:V_A(B')]_R=[V:V_A(B')]_L$.

The next is a partial extension of [9, Th. 3 (b)].

Theorem 4. *Let a division ring A be Galois over B, and $[V:C_0]<\infty$. If B' is an intermediate ring of A/B left finite over B then $[B':B]_L=[B':B]_R$.*

Proof. Since $[V:V_A(B')]_R\leqslant[B':B]_L<\infty$, we obtain $[V:V_A(B')]_R=[V:V_A(B')]_L$ by Prop. 4. Hence, by Cor. 3, $[B':B]_L=[B':B]_R$.

4. The notion of QG-1 extensions

Let $\mathfrak{H}$ be a (multiplicative) sub-semigroup of $\mathfrak{A}=\mathrm{Hom}\,(A, A)$.

Definition 1. $\mathfrak{H}$ is said to be *a left (resp. right) Galois semigroup* of A/B if $V_{\mathfrak{A}}(\mathfrak{H})\cap A_L=B_L$ and $V_{\mathfrak{A}}(\mathfrak{H})\cap A_R\supset B_R$ (resp. $V_{\mathfrak{A}}(\mathfrak{H})\cap A_L\supset B_L$ and $V_{\mathfrak{A}}(\mathfrak{H})\cap A_R=B_R$). If $\mathfrak{H}$ is a left and right Galois semigroup of A/B, we call $\mathfrak{H}$ *a Galois semigroup* of A/B.

Definition 2. Let $\mathfrak{H}$ be a left (resp. right) Galois semigroup of A/B. If A is $B\cdot V$-A-irreducible and $\mathfrak{H}A_R$ is a subring of $\mathfrak{A}$ (resp. A is A-$B\cdot V$-irreducible and $\mathfrak{H}A_L$ is a subring of $\mathfrak{A}$), then we call A/B *a left (resp. right) QG-1 extension with respect to* $\mathfrak{H}$, and $\mathfrak{H}$ a left (resp. right) Galois semigroup belonging to the left (resp. right) QG-1 extension A/B. Moreover, if A/B is a left and right QG-1 extension with respcet to a Galois semigroup $\mathfrak{H}$ then we call A/B *a QG-1 extension with respect to* $\mathfrak{H}$.

Finally we shall present a proposition which is useful in our consideration.

Proposition 6. *Assume that A/B is a left QG-1 extension w. r. t. a left Galois semigroup $\mathfrak{H}$ containing $\tilde{V}$. Let M be a B-B-submodule of A possessing a linearly independent finite left B-basis.*

(a) *$(\mathfrak{H}|M)V_R$ possesses a linearly independent V_R-basis that forms at the same time a linearly independent A_R-basis of $(\mathfrak{H}|M)A_R$, so that there holds $[M:B]_L=[(\mathfrak{H}|M)A_R:A_R]_R=[(\mathfrak{H}|M)V_R:V_R]_R$.*

(b) *Let B be a division ring, and a a non-zero element of M. The right V-module $a\mathfrak{H}V_R$ possesses a linearly independent right V-basis. In particular, if $[M:B]_L=[a\mathfrak{H}V_R:V]_R$ then $M=BaB$.*

Proof. (a) is essentially [9, Lemma 6], and the argument used in the proof of [6, Lemma 2.2] enables us to obtain the remaining.

5. Left QG-1 extensions of division rings

In the present section, we shall state without proof the extensions of [4, Th. 1] and [4, Th. 2] to left QG-1 extensions of division rings. In fact, by the validity of Prop. 6 and [6, Lemma 6.6], the respective proofs of them proceed in the same way as those of [4, Th. 1] (and [4, Cor. 2]) and [4, Th. 2] did.

Theorem 5. *Let a division ring A be a left QG-1 extension of B, and $[B:Z]=\infty$. If M is a B-B-submodule of A left finite over B then $M=BaB$ for some $a\in M$.*

Theorem 6. *Let a division ring A be a left QG-1 extension of B, and $[B:Z]=\infty$. If A/B is left algebraic then it is left locally finite.*

Remark. In [5, Th. 1], we have seen that if A/B is Galois and left algebraic and of bounded degree then $[A:B]<\infty$. However, the following example shows that [5, Th. 1] can not be extended to QG-1 extensions. Let $\Phi=\mathrm{GF}(p)$ (p a prime), and $x, y, z, \cdots$ an infinite number of indeterminates. If $B=\Phi(x, y, z, \cdots)$ and $A=\Phi(x^{1/p}, y^{1/p}, z^{1/p}, \cdots)$ then A/B is not finite dimensional but algebraic and of bounded degree.

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