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ON ADDITIVE FUNCTIONALS OF MEASURABLE FUNCTION SPACES

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1. Introduction

Let Ω be a measure space with finite measure μ. Recently, N. Friedman and M. Katz (1) discussed the representation of the additive functionals on function space $L_p(\Omega)$ ($p>0$) and showed that every additive functional Φ on L_p satisfying the continuity condition can be represented by the kernel function $K(h, \omega)$ as follows:

$$\Phi(f)=\int K(f(\omega), \omega)d\mu \qquad \text{for } f\in L_p.$$

In this paper, we discuss the representation theorems of additive functionals of more general measurable function space L_ρ and the structure of $K(h, \omega)$ will be deeply related to the function space L_ρ, provided that the measure space Ω is non-atomic with respect to μ. In fact, in the case $L_\rho=L_p(\mu)(p>0)$, we have found that $\Phi(f)=\int K(f(\omega), \omega)\, d\mu(\omega)$ is an additve functional of $L_p(\mu)$ iff $|K(h, \omega)|\leqq a(\omega)+A|h|^p$ for a. e. $\omega\in\Omega$ and every real number h, where A is a positive number and $0\leqq a(\omega)\in L_1(\mu)$.

The proofs of theorems in this paper owe to the idea of the Shimogaki's paper (2), even if his method is related to the theory of vector lattice. In this paper, we discuss only by measure theoritic method.

2. Function spaces L_ρ

Let Ω be a non-atomic measure space with finite measure μ and let $\rho(h, \omega)$ be a function of two variables $h\in(-\infty, \infty)$ and $\omega\in\Omega$, satisfying the following conditions:

(ρ. 1) $+\infty\geqq\rho(h, \omega)=\rho(-h, \omega)\geqq 0$, $\rho(0, \omega)=0$ for a. e. $\omega\in\Omega$.

(ρ. 2) $\rho(h, \omega)$ is a measusable function of ω for a fixed $h\in(-\infty, \infty)$.

(ρ. 3) $\rho(h, \omega)$ is a left-hand continuous function of $h\geqq 0$ for a. e. $\omega\in\Omega$.

(ρ. 4) $\rho(h_1, \omega)\geqq\rho(h_2, \omega)$ for $|h_1|\geqq|h_2|$.

We shall define a measurable function space L_ρ as follows:

$$L_\rho=\{f\,;\ \mu\text{-measurable and } \rho(f)=\int\rho(f(\omega), \omega)d\mu(\omega)<+\infty\}.$$

We see easily:

(1) $|f_1| \geqq |f_2|$ implies $\rho(f_1) \geqq \rho(f_2)$;

(2) if $L_\rho \in f$, $|f| \geqq |g|$ and g is measurable, then $g \in L_\rho$;

(3) L_ρ is a convex set and f, $g \in L_\rho$ implies $f+g \in L_\rho$;

(4) L_ρ is a lattice by the usual lattice operation;

(5) $0 \leqq f_i \uparrow f$ and $f_i, f \in L_\rho (i=1, 2, \cdots)$ imply $\rho(f) = \sup_i \rho(f_i)$;

(6) if $f_i (i=1, 2, \cdots)$ are measurable and mutually orthogonal, then

$$\rho\,(\sum_{i=1}^{\infty} f_i) = \sum_{i=1}^{\infty} \rho(f_i).$$

In the following, if necessary, a function $f \in L_\rho$ is identified with a function $g \in L_\rho$ which is equal to f except measure zero set. We shall state here the generalized Lebesque theorem which is need in the proof of Theorem 7.

Let $0 \leqq f_\lambda \in L_\rho$ *be a directed system with* $\sup_\lambda \rho(f_\lambda) < +\infty$, *then* $\sup_\lambda f_\lambda = f \in L_\rho$ *exists and* $\rho(f) = \sup_\lambda \rho(f_\lambda)$.

3. Additive functionals

A functional Φ on L_ρ is called an *additive functional* on L_ρ if

i) $|\Phi(f)| < +\infty$ for all $f \in L_\rho$;

ii) $\Phi(f+g) = \Phi(f) + \Phi(g)$ for $f, g \in L_\rho$ with $|f| \cap |g| = 0$.

By definition of additive functionals, we find the following properties of Φ:

(1) $\Phi(0) = 0$

(2) $\Phi(|f|) = \Phi(f^+) + \Phi(f^-)$ for $f \in L_\rho$, where $|f| = f^+ + f^-$, $f^+ = f \cup 0$, $f^- = (-f) \cup 0$.

If Φ satisfies

(3) $\Phi(-f) = -\Phi(f)$,

then Φ is called an *odd additive functional.*

Since L_ρ is not a linear space in general, it is not necessary that $f \in L_\rho$ implies $\alpha f \in L_\rho$ for all real number α.

Now, we shall consider the continuity conditions of additive functionals.

(CI) *For mutually orthogonal functions* $f_n \in L_\rho (n=1, 2 \cdots)$ *wtih* $\sum_{n=1}^{\infty} f_n \in L_\rho$, *it yields* $\Phi(\sum_{i=1}^{\infty} f_i) = \sum_{i=1}^{\infty} \Phi(f_i)$ (i.e. *absolutely convergent*).

(CII) $0 \leqq \alpha_i \uparrow \alpha$ *and* $\alpha f \in L_\rho$ *implies* $\Phi(\alpha_i f) \to \Phi(\alpha f)$.

For an additive functional Φ on L_ρ, we can define a new additive functional such that

$$|\Phi|(f)=\sup_{0\leq|g|\leq|f|}|\Phi(g)|.$$

Clearly $|\Phi|$ satisfies the following conditions:

(i) $|\Phi|(0)=0$

(ii) $|\Phi|(f)\geqq 0$ for all $f\in L_\rho$.

(iii) $|\Phi|(f+g)=|\Phi|(f)+|\Phi|(g)$ for $|f|\cap|g|=0$.

(iv) $|\Phi|(f)=|\Phi|(|f|)$.

(v) $|f|\leqq|g|$ implies $|\Phi|(f)\leqq|\Phi|(g)$.

It may happen that $|\Phi|(f)=+\infty$ for some $f\in L_\rho$. If Φ satisfies that $|\Phi|(f)<\infty$ for all $f\in L_\rho$, then Φ is called a type of *bounded variation.*

Theorem 1. *If an additive functional Φ satisfies* (CI), *then Φ is a type of bounded variation.*

Proof. Suppose that $|\Phi|(f)=+\infty$ for some $f\in L_\rho$. Since the measure space Ω is non-atomic, we find a sequence of measurable sets $\Omega\supset e_1\supset e_2\supset$ ……such that $\mu(e_n)\to 0$ and $|\Phi|(f\cdot\chi e_n)=+\infty$ for $n=1,2,\cdots$, where χe_n is a characteristic function on a measurable set e_n.

Since Φ satisfies (CI) and $\mu(e_n)\to 0$, we find that $\Phi(g\chi e_n)\to 0$ for all $g\in L_\rho$.

By induction, if we define a function $f_n\in L_\rho$ with $|f_n|\leq|f|$, $f_n=f_n\cdot\chi e_{i_n}$ and $|\Phi(f_n)|>n$, then we can find a number i_{n+1} such that $g_n=f_n-f_n\chi e_{i_{n+1}}$ and $|\Phi(g_n)|>n/2$, since $\Phi(g\chi e_n)\to 0$ for all $g\in L_\rho$ by (CI). Hence we find a function $f_{n+1}=f_{n+1}\chi e_{i_{n+1}}$ with $|\Phi(f_{n+1})|>n+1$ and $|f_{n+1}|\leq|f|$ because of $|\Phi|(f\chi e_n)=+\infty$.

By the considerations above, for all natural numbers $n\geqq 1$, we find a function $g_n=f_n-f_n\chi e_{i_{n+1}}$ with $|g_n|\leqq|f|$ and $|\Phi(g_n)|>n/2$. Clearly the sequence of functions $\{g_n\}$ is mutually orthogonal. Since $|\sum_{n=1}^\infty g_n|\leqq|f|$, we have $\sum_{n=1}^\infty g_n\in L_\rho$. But $\Phi(\sum_{n=1}^\infty g_n)=\sum_{n=1}^\infty\Phi(g_n)$ is not convergent. This is a contradiction.

Theorem 2. *If Φ satisfies* (CI), *then $|\Phi|$ is also an additive functional satisfying* (CI).

Proof. Let $\{f_i\}$ be a sequence of orthogonal functions with $\sum_{i=1}^\infty f_i=f\in L_\rho$. For every $\varepsilon>0$, there exists $|g_\varepsilon|\leqq|f|$ with $||\Phi|(f)-|\Phi(g_\varepsilon)||\leqq\varepsilon$. Let $e_i=\{x;\ f_i(x)\neq 0\}$. By (CI), we have $\Phi(g_\varepsilon)=\sum_{i=1}^\infty\Phi(g_\varepsilon\chi e_i)$. Hence, $|\Phi(g_\varepsilon)|$ $=\sum_{i=1}^\infty|\Phi(g_\varepsilon\chi e_i)|\leqq\sum_{i=1}^\infty|\Phi|(f_i)\leqq|\Phi|(f)$ i.e. $|\sum_{i=1}^\infty|\Phi|(f_i)-|\Phi|(f)|\leqq\varepsilon$. Since

ε is arbitrary, we see $|\Phi|(f)=\sum_{i=1}^{\infty}|\Phi|(f_i)$.

Theorem 3. *If Φ satisfies* (CII), *then $|\Phi|$ is an additive functional satisfying* (CII).

Proof. By theorem 1, we have $|\Phi|(f)<+\infty$ for all $f\in L_\rho$. Let $0\leqq\alpha_i\uparrow\alpha$ and $\alpha f\in L_\rho$. For any positive number $\varepsilon>0$, there exists f_ε such that $|f_\varepsilon|\leq|\alpha f|$ and $||\Phi|(\alpha f)-|\Phi(f_\varepsilon)||<\varepsilon$. Since $\frac{\alpha_i}{\alpha}\to 1$, $|\Phi(\frac{\alpha_i}{\alpha}f_\varepsilon)-\Phi(f_\varepsilon)|<\varepsilon$ for large i.

$$\begin{aligned}
0&\leqq|\Phi|(\alpha f)-|\Phi|(\alpha_i f)\\
&\leqq|\Phi|(\alpha f)-|\Phi|(\frac{\alpha_i}{\alpha}f_\varepsilon)\\
&\leqq|\Phi|(\alpha f)-\Phi(\frac{\alpha_i}{\alpha}f_\varepsilon)\\
&=|\Phi|(\alpha f)-\Phi(f_\varepsilon)+\Phi(f_\varepsilon)-\Phi(\frac{\alpha_i}{\alpha}f_\varepsilon)\\
&<2\varepsilon.
\end{aligned}$$

Hence, we have

$$|\Phi|(\alpha f)=\lim_{i\to\infty}|\Phi|(\alpha_i f).$$

We shall consider another condition of continuity.

(CIII) $0\leqq f_i\uparrow f$, *or* $0\geqq f_i\downarrow f$ *with* $f\in L_\rho$ *imply* $\Phi(f_i)\to\Phi(f)$.

Clearly, (CIIII) implies (CI) and (CII).

Theorem 4. *If Φ satisfies* (CI) *and* (CII), *then $|\Phi|$ satisfies* (CIII).

Proof. Let $0\leqq f_i\uparrow f\in L_\rho$, For any $1>\varepsilon>0$, we shall consider the function

$$(f_i-(1-\varepsilon)f)\cup 0=g_i.$$

For the set $e_i=\{x\,;g_i(x)>0\}$, we have

$e_i\subset e_{i+1}\subset\cdots$ and $\lim_{i\to+\infty}e_i=e=\{x\,;f(x)\neq 0\}$.

Since $(1-\varepsilon)f\cdot\chi e_i\leqq f_i\chi e_i\uparrow f$, we have

$$|\Phi|((1-\varepsilon)f\cdot\chi e_i)\leqq|\Phi|(f_i\chi e_i)\leqq|\Phi|(f_i)\leqq|\Phi|(f).$$

and so,

$$|\Phi|((1-\varepsilon)f)\leqq\sup_i|\Phi|(f_i)\leqq|\Phi|(f).$$

Since ε is arbitrary positive number, we see

$$\sup_i|\Phi|(f_i)=|\Phi|(f).$$

In the proof just above, we use the *monotone property*:

$$|f|\leqq|g|\ \textit{implies}\ |\Phi|(f)\leqq|\Phi|(g).$$

Hence, for arbitrary monotone additive functional Φ, (i.e. $|f|\leqq|g|$ implies $\Phi(f)\leqq\Phi(g)$), we have

Theorem 5. *If Φ is monotone and Φ satisfies* (CI) *and* (CII), *then Φ satisfies* (CIII).

Remark. There exists an additive functional which satisfies (CI) and (CII) and is not monotone, but does not satisfy (CIII).
We shall show such example.
Let $\rho(h,t)=h$ for all $-\infty<h<\infty$. $0<t<1$ and let μ be usual Lebesque measure on (0, 1) then $L_\rho=L_1(0,1)$.
For $f\in L_1(0,1)$, we put $e_\lambda=\{x\,;f(x)=\lambda x\}$, then $e_{\lambda,\lambda'}=e_\lambda\cap e_{\lambda'}$ is a measure 0 set for $\lambda\neq\lambda'$. Hence, e_λ which has positive measure, is at most countable. We denote e_f by the union of such measurable sets. We shall consider an additive functional Ψ defined by $\Psi(f)=\int_{(0,1)-e_f}fd\mu$.
Then, Ψ satisfies (CI) and (CII), but not (CIII).

Theorem 6. *Let Φ be an additive functional on L_ρ which satisfies* (CI). *Then there exist positive numbers ε, $\delta>0$ such that $\rho(f)\leqq\varepsilon$ implies $|\Phi(f)|\leqq\delta$.*

Proof. By Theorem 2, $|\Phi|$ satisfies (CI) if Φ satisfies (CI).
Since $|\Phi|(f)\geqq|\Phi(f)|$ for every $f\in L_\rho$, we shall prove that there exist ε, $\delta>0$ such that $\rho(f)\leqq\varepsilon$ implies $|\Phi|(f)\leqq\delta$. If there exists no such positive number, then we can find an positive measurable function $0\leqq f_n\in L_\rho$ such that

$$\rho(f_n)\leqq\frac{1}{2^n}\ \text{and}\ |\Phi|(f_n)\geqq n\ \text{for every}\ n=1,2,\cdots.$$

$\rho(f_1\cup\cdots\cup f_n)\leqq\rho(f_1)+\rho(f_2)\cdots+\rho(f_n)\leqq1$ and the sequence of functions $g_n=f_1\cup f_2\cup\cdots\cup f_n$ is a monotone increasing system with $\rho(g_n)\leqq1$.
Hence, there exists $f=\bigcup_{n=1}^{\infty}f_n=\bigcup_{n=1}^{\infty}g_n\in L_\rho$.
Since $|\Phi|(f)\geqq|\Phi|(f_n)\geqq n$ for all $n=1,2,\cdots$, we have $|\Phi|(f)=+\infty$.

4. Representation of additive functionals

Let Φ be an additive functional on L_ρ. We shall assume here that Φ satisfies (CIII). Let χe be a characteristic function on e (μ-measurable subset of Ω). For every positive number $h\geqq0$, $\Phi(h\chi e)=\mu_h(e)$, (provided that $h\chi e\in L_\rho$,) is a measure of bounded variation on the measure space e_h (There exists a maximal measurable set e_h so that $\Phi(h\chi e_h)<+\infty$). $\mu_h(e)$ is absolutely continuous with respect to μ. Hence, by Radon-Nikodym's theorem, we find an integrable function $K(h,\omega)$ such that

$$\mu_h(e)=\int_e K(h,\ \omega)\,d\mu$$

for all measurable subset e of e_h.

For $\omega\in\Omega-e_h$, we define $K(h,\omega)=\infty$. We see also that $\rho(h,\omega)=\infty$ a. e. $\omega\in\Omega-e_h$. Let $\{h_i\}$ be a system of whole positive rational numbers. For every $h>0$, we can choose h_i with $h_i\uparrow h$. Since $\int_e|K(h_i,\ \omega)|\,d\mu\leqq |\Phi|(h_i\chi e)\leqq|\Phi|(h\chi e)$ for all measurable set $e\subset e_h$, applying Radon-Nykodym's theorem to $|\Phi|(h\chi e)$, we see that $|K(h_i,\omega)|$ is bounded by integrable function.

By virtue of Lebesque majorant theorem and (CIII), we have

$$(*)\qquad \lim_{i\to\infty}K(h_i,\omega)=K(h,\omega)\ \text{ a. e. } \omega\in\Omega.$$

For every negative $h<0$, we find also $K(h,\omega)$ such that $h_i\downarrow h$ implies $K(h_i,\omega)\to K(h,\omega)$ a. e. $\omega\in\Omega$.

We define $K(h,\omega)$ as $(*)$.

The function $K(h,\omega)$ of two variables h,ω is measurable with respect to ω for fixed h, and left-hand continuous with respect to h for a fixed ω (a. e. ω of Ω), if $h>0$ and $K(h,\omega)$ is finite. By (CIII) and additivity of Φ, using Lebesque majorant theorem, we have

$$\Phi(f)=\int_\Omega K(f(\omega),\omega)d\mu=\lim_{i\to\infty}\int K(f_i(\omega),\omega)d\mu(\omega)$$

for $0\leqq f_i\uparrow f\in L_\rho$, where f_i is a step function for $i=1,2,\cdots$. For $L_\rho\ni f\leqq 0$, where $f_i\downarrow f(f_i$ is a step function for $i=1,2,\cdots)$, we have

$$\Phi(f)=\int_\Omega K(f(\omega),\omega)\,d\mu=\lim_{i\to\infty}\int K(f_i(\omega),\omega)d\mu.$$

For arbitrary $f\in L_\rho$, we decompose f into $f=f^+-f^-$ with $f^+,f^-\in L_\rho$, $f^+\cap f^-=0$.

$$\Phi(f)=\Phi(f^+)+\Phi(-f^-)=\int_\Omega K(f^+(\omega),\omega)d\mu+\int_\Omega K(-f^-(\omega),\omega)d\mu.$$

Hence we have the following theorem :

Theorem 7. *Let Φ be an additive functional on L_ρ, and satisfy* (CIII). *Then we find a kernel function $K(h,\omega)$ which satisfies*:

(1) *measurable with respect to $\omega\in\Omega$ for a fixed h.*

(2) $0\leqq h_i\uparrow h$ *or* $0\geqq h_i\downarrow h$, *then* $K(h_i,\omega)\to K(h,\omega)$ *for a. e.* $\omega\in\Omega$ (*provided that $K(h,\omega)$ is finite*) *such that*

$$\Phi(f)=\int_\Omega K(f(\omega),\omega)d\mu\quad\text{for}\quad f\in L_\rho,$$

where $|K(h,\omega)|<+\infty$ *if* $\rho(h,\omega)<+\infty$.

Remark. If Φ is odd, then the kernel function $K(h, \omega)$ is an odd function with respect to h (i. e. $K(h, \omega) = -K(-h, \omega)$) for a. e. $\omega \in \Omega$.

Theorem 8. *There exists a positive number $A > 0$ and $0 \leqq s(\omega) \in L_1(\mu)$ such that*

$$(\ddagger) \qquad |K(h, \omega)| \leqq s(\omega) + A\rho(h, \omega).$$

Proof. We define a functional T on L_ρ as follows:

$$T(f) = \int_\Omega \{ |K(f(\omega), \omega)| - A\rho(f(\omega), \omega)\} d\mu$$

where $A > \delta/\varepsilon$ (ε, δ are the positive numbers determined by Theorem 6.) We shall show that

$T(f(\omega), \omega) \geqq 0$ a. e. $\omega \in \Omega$ implies $\rho(f) < \varepsilon$, where $T(h, \omega) = |K(h, \omega)| - A\rho(h, \omega)$.

If not, there exists $f(\omega) \in L_\rho$ such that $T(f(\omega), \omega) \geqq 0$ for a. e. $\omega \in \Omega$ and $\int \rho(f(\omega), \omega) d\mu > \varepsilon$.

Since Ω is non-atomic, we can find a measurable set e with

$\int_e \rho(f(\omega), \omega) d\mu = \rho(f \cdot \chi e) = \varepsilon$. Clearly $T(f \cdot \chi e(\omega), \omega) \geqq 0$ a. e. $\omega \in \Omega$. By theorem 6,

$$\begin{aligned} \delta \geqq |\Phi|(f \cdot \chi e) &\geqq \int |K(f \cdot \chi e(\omega), \omega)| d\mu \\ &= \int T(\mathrm{X} \cdot \chi e(\omega), \omega) d\mu + A \int \rho(f \cdot \chi e(\omega), \omega) d\mu \\ &= \int T(f \cdot \chi e(\omega), \omega) d\mu + A\rho(f \cdot \chi e(\omega)) \\ &\geqq A\rho(f \cdot \chi e) > \frac{\delta}{\varepsilon} \cdot \varepsilon = \delta. \end{aligned}$$

This is a contradiction.

The system of measurable functions $f(\omega) \in L_\rho$ with $T(f(\omega), \omega) \geqq 0$ a. e. $\omega \in \Omega$ is a directed system. In fact, $T(f_1(\omega), \omega) \geqq 0$ and $T(f_2(\omega), \omega) \geqq 0$ a. e. $\omega \in \Omega$, then

$$T(f_1 \cup f_2(\omega), \omega) = \begin{cases} T(f_1(\omega), \omega) \geqq 0 & \text{for } \omega \text{ with } f_1(\omega) \geqq f_2(\omega); \\ T(f_2(\omega), \omega) \geqq 0 & \text{for } \omega \text{ with } f_2(\omega) \geqq f_1(\omega). \end{cases}$$

By generalized Lebesque theorem, we find a measurable function

$$s(\omega) = \operatorname*{ess\text{-}sup}_{T(f(\omega), \omega) \geqq 0} f(\omega) \in L^1(\mu).$$

Hence, we have

$$|K(f(\omega), \omega)| \leqq s(\omega) + A \cdot \rho(f(\omega), \omega) \quad \text{for a. e. } \omega \in \Omega.$$

Considering $f(\omega) = h \cdot \chi_e(\omega)$ for arbitrary measurable set $e \subset \Omega$, we have

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$$|K(h, \omega)| \leqq s(\omega) + A \cdot \rho(h, \omega) \quad \text{for a. e. } \omega \in \Omega.$$

We see easily,

Theorem 9. *If $K(h, \omega)$ satisfies the conditions* (1), (2) *of Theorem* 7 *and* (⁑) *of Theorem* 8, *then*

$$\Phi(f) = \int_\Omega K(f(\omega), \omega) d\mu \quad \text{for } f \in L_\rho.$$

is an additive functional which satisfies (CIII).

By the method used in the proof of Theorem 8, we find that:

Theorem 10. *If Φ satisfies* (CI), *then we find positive numbers* A, B *such that*

$$|\Phi(f)| \leqq B + A\rho(f) \quad \text{for } f \in L_\rho.$$

5. Supplementary results

In some cases, we must consider linear spaces. If we condider $\widetilde{L}_\rho$: the totality of μ-measurable functions with $\rho(\alpha f) < +\infty$ for some $\alpha \neq 0$, then $\widetilde{L}_\rho$ is a linear space.

In the following, we shall consider an additive functional Φ *on* $\widetilde{L}_\rho$.

At first, we have

Theorem 11. $L_\rho = \widetilde{L}_\rho$ *(i. e. L_ρ is a linear space) if and only if there exist positive numbers* A *and* $0 \leqq s(\omega) \in L_1(\mu)$ *such that*

$$\rho(2h, \omega) \leqq s(\omega) + A\rho(h, \omega) \quad a.\ e.\ \omega \in \Omega.$$

The proof is the same as the proof of Theorem 8. This theorem is also obtained by Shimogaki.

Theorem 12. *If an addtive functional on $\widetilde{L}_\rho$ satisfies* (CIII), *then there exists a function $K(h, \omega)$ of two variables which satisfies the conditions* (1), (2) *of Theorem* 7, *such that*

$$\Phi(f) = \int_\Omega K(f(\omega), \omega) d\mu$$

and for every $n = \pm 1, \pm 2, \cdots\cdots$, there exists a positive number $A_n > 0$ and $s(\omega) \in L_1(\mu)$ such that

$$|K(nh, \omega)| \leqq s(\omega) + A_n \rho(h, \omega)$$

for all h and a. e. $\omega \in \Omega$.

The proof of this theorem is as same as the proof of Theorem 7 and Theorem 8, and so it is omitted.

The continuity condition of additive functionals on $L_p(p > 0)$ considered by Friedman and Katz is equivalent to the so-called order-continuous condition:

(CIV) $f_i(\omega) \to f(\omega)$ *a. e.* $\omega \in \Omega$ *and* $\bigcup_{i=1}^{\infty} |f_i| \in L_\rho$ *implies* $\Phi(f_i) \to \Phi(f)$.

The condition (CIV) implies (CI), (CII) and (CIII).

Let Φ be an additive functional on L_ρ satisfying (CIV). Then the kernel function $K(h, \omega)$ which is defined in Theorem 7 is continuous with respect to h for a. e. $\omega \in \Omega$.

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