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SOME INTEGRAL FORMULAS FOR RIEMANNIAN MANIFOLDS

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1. Introduction. Let (M, g) be a compact n -dimensional Riemannian manifold with positive definite metric tensor $g = (g_{ij})$ and ∇ the Levi-Civita connection. By $R = (R_{hij}{}^k)$, $\rho(R) = (R_{ij})$ and $\tau(R) = R$, we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively.

In his paper [2], T. Sakai obtained the following integral formula :

$$(1.1) \quad \int_M K dM = \int_M (\|(\nabla\tau)(R)\|^2 - 4\|(\nabla\rho)(R)\|^2 + \|\nabla R\|^2) dM,$$

where dM is the volume element of (M, g) and

$$(1.2) \quad \begin{aligned} K = & 4 R^{hi} R_h{}^a R_{ia} - 4 R^{hi} R^{jk} R_{hjk i} - 2 R^{ab} R_{ahij} R_b{}^{hi j} \\ & - 4 R^{hijk} R^a{}_{hj}{}^b R_{aikb} - R^{hijk} R_{hiab} R_{jk}{}^{ab}. \end{aligned}$$

In this note, we define a generalized curvature tensor field $T = (T_{hijk})$ by

$$(1.3) \quad \begin{aligned} T_{hijk} = & -\frac{1}{4} (R_h{}^a R_{aijk} + R_i{}^a R_{haji} + R_j{}^a R_{hika} + R_k{}^a R_{hija}) \\ & + R^a{}_{hi}{}^b R_{jkab} - R^a{}_{hj}{}^b R_{aikb} + R^a{}_{ij}{}^b R_{ahkb}. \end{aligned}$$

where $T_{hijk} = T_{hij}{}^a g_{ak}$. Then, as for T , we shall prove the following theorems.

Theorem A. *Let (M, g) be a compact Riemannian manifold and C the 3-dimensional Weyl's conformal curvature tensor field. Then the following formula holds :*

$$2 \int_M (C, T) dM = \int_M (\|(\nabla\tau)(R)\|^2 - 4\|(\nabla\rho)(R)\|^2 + \|\nabla R\|^2) dM.$$

Theorem B. *Let (M, g) be a locally symmetric Riemannian manifold. Then the generalized curvature tensor field T is identically zero.*

Theorem C. *Let (M, g) be a compact Riemannian manifold. If the Ricci curvature tensor satisfies the Codazzi equation $\nabla_h R_{ij} = \nabla_i R_{hj}$, then we*

have

$$\int_M (R, T) dM \geq 0.$$

The equality holds on M if (M, g) is a locally symmetric Riemannian manifold.

2. Notations and definitions. We denote the contravariant components of g by g^{ij} and follow the summation convention for repeated indices. Let $(\cdot, \cdot)$ denote inner product and $\|\cdot\|$ norm.

The Riemannian curvature tensor $R = (R_{hij}{}^k)$ is defined by $R(X, Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X, Y]}Z$ for vector fields X, Y and Z and $R(\partial_h, \partial_i)\partial_j = R_{hij}{}^k\partial_k$. Let us denote the covariant components of R by $R_{hijk} = g(R(\partial_h, \partial_i)\partial_j, \partial_k)$. Then the components of the Ricci curvature tensor $\rho(R)$ and the scalar curvature $\tau(R)$ are given, respectively, by $R_{ij} = g^{ab}R_{aibj}$ and $R = g^{ab}R_{ab}$.

A $(1, 3)$ -tensor field $D = (D_{hij}{}^k)$ is called a generalized curvature tensor field if it satisfies the conditions:

$$(2.1) \quad \begin{aligned} (1) \quad & D_{hij}{}^k = -D_{ihj}{}^k, \\ (2) \quad & D_{hijk} = -D_{hikj} \quad (D_{hijk} = D_{hij}{}^a g_{ak}), \\ (3) \quad & D_{hij}{}^k + D_{ijh}{}^k + D_{jhi}{}^k = 0 \end{aligned}$$

We denote the Ricci curvature tensor and the scalar curvature associated with D by $\rho(D) = (D_{ij}) = (g^{ab}D_{aibj})$ and $\tau(D) = D = g^{ab}D_{ab}$, respectively. By virtue of the first Bianchi identity, we can find that T has the properties (1) ~ (3) in (2.1). The Ricci curvature tensor $\rho(T) = (T_{ij})$ and the scalar curvature $\tau(T) = T$ associated with the tensor field T are given, respectively, by

$$(2.2) \quad T_{ij} = \frac{1}{2}(R^{ab}R_{aibj} - R_i{}^a R_{aj}), \quad T = 0.$$

If we define the $(1, 3)$ -tensor fields $E = (E_{hij}{}^k)$ and $F = (F_{hij}{}^k)$, in the covariant form, by

$$(2.3) \quad \begin{aligned} E_{hijk} &= -\frac{1}{4}(R_h{}^a R_{aibj} + R_i{}^a R_{hajk} + R_j{}^a R_{hia k} + R_k{}^a R_{hija}), \\ F_{hijk} &= R_{hi}{}^a R_{jakb} - R_{hj}{}^a R_{aikb} + R_{ij}{}^a R_{ahkb}, \end{aligned}$$

then the tensor fields E and F are also generalized curvature tensor fields. Obviously, $T = E + F$. The Ricci curvature tensors $\rho(E) = (E_{ij})$ and $\rho(F) = (F_{ij})$ are given, respectively, by

$$(2.4) \quad \begin{aligned} E_{ij} &= -\frac{1}{2} (R^{ab} R_{aijb} + R_i^a R_{aj}), \\ F_{ij} &= R^{ab} R_{aijb}, \end{aligned}$$

and the scalar curvatures $\tau(E) = E$ and $\tau(F) = F$ are given by

$$(2.5) \quad F = -E = R^{ab} R_{ab}.$$

3. Proof of the theorems. Transvecting (1.3) with R^{hijk} and using the first Bianchi identity, we obtain

$$(3.1) \quad \begin{aligned} 2(T, R) &= -R^{hijk} R_{hlab} R_{jk}^{ab} - 4R^{hijk} R_{ahjb} R_{ik}^{ab} \\ &\quad - 2R^{ab} R_{aijk} R_b^{ijk}, \end{aligned}$$

and transvecting (2.2) with R^{ij} , we have

$$(3.2) \quad 2(\rho(T), \rho(R)) = R^{hi} R^{jk} R_{hjk i} - R_i^a R_{aj} R^{ij}.$$

Substituting (3.1) and (3.2) into (1.2), we can find that K is written as

$$(3.3) \quad K = 2(T, R) - 8(\rho(T), \rho(R)).$$

On the other hand, the covariant components of the 3-dimensional Weyl's conformal curvature tensor field $C = (C_{hijk})$ are given by

$$(3.4) \quad \begin{aligned} C_{hijk} &= R_{hijk} - (g_{hk} R_{ij} - g_{ik} R_{hj} + R_{hk} g_{ij} - R_{ik} g_{hj}) \\ &\quad + \frac{R}{2} (g_{hk} g_{ij} - g_{ik} g_{hj}) \end{aligned}$$

Transvecting (3.4) with T^{hijk} and using (2.2), we have

$$(3.5) \quad (C, T) = (R, T) - 4(\rho(R), \rho(T)).$$

By (3.3), K is rewritten as $K = 2(C, T)$. Thus, we obtain Theorem A.

Next, we define a $(0, 6)$ -tensor field $H = (H_{hijk, pq})$ by

$$(3.6) \quad H_{hijk, pq} = -(\nabla_p \nabla_q R_{hijk} - \nabla_q \nabla_p R_{hijk}).$$

Then, $H_{hijk, pq}$ is expressed in the form

$$(3.7) \quad \begin{aligned} H_{hijk, pq} &= R_{pqh}^a R_{aijk} + R_{pqi}^a R_{hajk} \\ &\quad + R_{pqj}^a R_{h i a k} + R_{pqk}^a R_{h i j a}. \end{aligned}$$

By contracting h with q in (3.7) and using (2.3), we have

$$(3.8) \quad H_{ijk, pa}^a = R_p^a R_{aijk} - F_{jkpi}.$$

If (M, g) is a locally symmetric manifold, then the equation $H = O$ holds. By (3. 8), we have $R_h^a R_{aijk} = F_{hijk}$. Substituting this equation into the first equation in (2. 3), we have $T = O$. Thus, we obtain theorem B.

Finally, we shall prove Theorem C. The following integral formula is well known (cf. [1]) :

$$(3. 9) \quad \begin{aligned} & 2 \int_M (\nabla_h R_{ij} - \nabla_i R_{hj})(\nabla^h R^{ij} - \nabla^i R^{hj}) - R^{hijk} H_{ijk, ha}^a dM \\ & = \int_M \|\nabla R\|^2 dM \end{aligned}$$

where dM is the volume element of (M, g) . On the other hand, by (3. 8), we have

$$(3. 10) \quad R^{hijk} H_{ijk, ha}^a = -R^{hijk} T_{hijk}.$$

Therefore, if the Ricci curvature tensor $\rho(R) = (R_{ij})$ satisfies the condition $\nabla_h R_{ij} = \nabla_i R_{hj}$, then the formula (3. 9) can be reduced to

$$(3. 11) \quad 2 \int (R, T) dM = \int_M \|\nabla R\|^2 dM.$$

Theorem C is an immediate consequence of (3. 11).

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