Mathematical Journal of Okayama University

Volume 27, *Issue* 1 1985 *Article* 9

JANUARY 1985

Rings decomposed into direct sums of nil rings and certain reduced rings

Yasuyuki Hirano[*] Hisao Tominaga[†]

[*]Okayama University
[†]Okayama University

Math. J. Okayama Univ. 27 (1985), 35—38

RINGS DECOMPOSED INTO DIRECT SUMS OF NIL RINGS AND CERTAIN REDUCED RINGS

Dedicated to Professor Noboru Ito on his 60th birthday

YASUYUKI HIRANO and HISAO TOMINAGA

Recently, in his paper [5], M. Ôhori introduced the concept of generalized right (resp. left) p.p. rings with identity. This concept can be extended to rings without identity as follows: An element x of a ring R is called a *right* (resp. *left*) *p. p. element* if there exists an idempotent e in R such that $xe=x$ and $r(x) = r(e)$ (resp. $ex=x$ and $l(x) = l(e)$), where $r(*)$ (resp. $l(*)$) denotes the right (resp. left) annihilator of $*$ in R. A ring R is called a *generalized right* (resp. *left*) *p. p. ring* if for every $x \in R$ there exists a positive integer n such that x^n is a right (resp. left) p. p. element, and R is a right (resp. left) p. p. ring if every $x \in R$ is a right (resp. left) p. p. element.

Obviously, every π-regular ring is a generalized (right and left) p. p. ring, and every direct sum of generalized right (resp. left) p.p. rings whose idempotents are central is also a generalized right (resp. left) p. p. ring. For instance, every (probably infinite) direct sum of domains with identity is a (right and left) p. p. ring.

Throughout the present paper, R will represent a ring. Let N be the set of nilpotent elements in R, and P the set of right p. p. elements in R. Given an integer $q>1$, we set $E_q = \{x \in R \mid x^q = x\}$; in particular, $E = E_2$.

We consider the following conditions:

($\sharp$) Each $x \in R$ has at most one representation of the form $x=x'+x''$, where $x' \in N$ and $x'' \in P$.

($\sharp$)′ Each $x \in R$ has at most one representation of the form $x=x'+x''$, where $x' \in N$ and x'' is right regular ($x''=x''^2y$ for some $y \in R$).

($\sharp$)″ Each $x \in R$ has at most one representation of the form $x=x'+x''$, where $x' \in N$ and x'' is potent ($x''=x''^k$ for some integer $k>1$).

($*$) E is contained in some reduced ideal A of R.

The purpose of this paper is to prove the following theorem which deduces numerous decomposition theorems, among others, [1, Theorem 3] and [6, Theorem 1].

Theorem 1. *The following conditions are equivalent*:

1) *R is a generalized right p. p. ring and satisfies* (♯).

2) *R is a generalized right p. p. ring and satisfies* ($*$).

3) *$R = N \oplus P$; strictly speaking, both N and P are ideals of R and R is the direct sum of N and P.*

When this is the case, P is a reduced (right and left) p. p. ring.

Proof. Obviously, 3) implies 1).

1) ⇨ 2). Let e be an arbitrary element of E. Given $x \in R$, we set $u = ex - exe$. Since $u^2 = 0$ and $e+u$ is also in E, (♯) implies that $u = 0$, i.e., $ex = exe$. Similarly, we can show that $xe = exe$, and therefore e must be central. Now, let $v \in N \cap eR$. Since $v+e$ is invertible in the ring eR, it is a right p. p. element of R; (♯) implies that $v = 0$. Hence eR is a reduced ring. Now, let A be the ideal of R generated by E. Since E is contained in the center of R, for any $a \in A$ there is some $f \in E$ such that $a \in fR$. Therefore, a cannot be a non-zero nilpotent element. Consequently, A is a reduced ideal.

2) ⇨ 3). As above, we can easily show that every idempotent of R is central. Furthermore, there holds $A = P$. Let x be an arbitrary element of R. Then, by hypothesis, there exists a positive integer m and an idempotent e such that $x^m e = x^m$ and $r(x^m) = r(e)$. Clearly, $(xe)e = xe$ and $r(xe) = r(e)$, and so $xe \in P$. Since e is central, we have $(x-xe)^m = 0$. Therefore, x is the sum of $x-xe \in N$ and $xe \in P$. Next, we claim that if $u \in N$ and $y \in R$ then yu and $uy \in N$. Actually, there exists a positive integer n and $f \in E$ such that $(yu)^n f = (yu)^n$ and $r((yu)^n) = r(f)$. Let k be the least positive integer such that $u^k f = 0$. If $k > 1$, then $(yu)^n u^{k-1} = (yu)^n f u^{k-1} = (yu)^{n-1} y u^k f = 0$, which forces a contradiction $fu^{k-1} = 0$. We conclude therefore that $(yu)^n = (yu)^n f = 0$ and $(uy)^{n+1} = 0$. In particular, we get $PN = 0 = NP$. Now, let $v, v' \in N$, and $v+v' = w+p$, where $w \in N$ with $w^l = 0$ and $p \in P$. In view of $NP = 0$, we get $(v+v')^2 = (v+v')(w+p) = (v+v')w$, and hence $(v+v')^{l+1} = (v+v')w^l = 0$. Thus we have shown that N forms an ideal of R and $R = N \oplus P$.

Corollary 1. *The following conditions are equivalent*:

1) *R is a π-regular ring and satisfies* (♯).

1)′ *R is a π-regular ring and satisfies* (♯)′.

2) *R is a π-regular ring and satisfies* $(*)$.
3) $R = N \oplus P$, *and P is a strongly regular ring.*

Proof. By Theorem 1 (and its proof), it suffices to show that 1)′ implies 2). Since we can show that every idempotent of R is central and R is strongly π-regular, the proof proceeds in the same way as in that of 1) ⇒ 2) of Theorem 1.

Now, let R be a P'_n-ring in the sense of [2], that is, $xR^n = xR^n x$ for all $x \in R$. Then $xR^n = xR^n x^k$ for $k = 1, 2, \cdots$; in particular, R is a left π-regular ring with $N^{n+1} = 0$. Hence, by a result of Zöschinger-Dischinger (see, e.g., [3, Proposition 2]), R is strongly π-regular. Furthermore, if $e \in E$ and $u^2 = 0$ then $ue \in uR^n u^2 = 0$ and $eu = e \cdot e^{n-1} u \in eR^n e$, whence we see that $eu = eue = 0 = ue$. This enables us to see that e is central. For any $x \in R$, we now have $ex = exe^n \in exR^n = exR^n(ex)^{n+1}$, which proves that eR is a reduced ideal of R. Hence, R satisfies $(*)$. This fact together with Corollary 1 gives the following which includes the main part of [2, Theorem 2].

Corollary 2. *The following conditions are equivalent:*
1) *R is a π-regular ring with $N^{n+1} = 0$ and satisfies* (♯).
1)′ *R is a π-regular ring with $N^{n+1} = 0$ and satisfies* (♯)′.
2) *R is a π-regular ring with $N^{n+1} = 0$ and satisfies* $(*)$.
3) $R = N \oplus P$, *P is strongly regular, and* $N^{n+1} = 0$.
4) *R is a P'_n-ring.*

In the same way as for Corollary 1, we can prove the following which includes [1, Theorem 3] and [6, Theorem 1].

Corollary 3. *The following conditions are equivalent:*
1) *R is a periodic ring and satisfies* (♯).
1)″ *R is a periodic ring and satisfies* (♯)″.
2) *R is a periodic ring and satisfies* $(*)$.
3) $R = N \oplus P$, *and P is a J-ring (every element of P is potent).*

Finally, we shall prove the following

Corollary 4. *If R is a ring with* 1, *then the following conditions are equivalent:*
1) *The addition "+" of R is equationally definable in terms of the*

Mathematical Journal of Okayama University, Vol. 27 [1985], Iss. 1, Art. 9

multiplication " · " *and the successor operation* "$\wedge$" *of* R, *and* R *satisfies* (♯) (*or* (♯)").

2) *There exists a positive integer* n *such that* $R = E_{n+1}$.

Proof. If R satisfies 1), then there exists a positive integer n such that $x^n = x^{2n}$ for all $x \in R$, by [4, Theorem 1]. Obviously, R is of bounded index at most n. Since $(x^{n+1})^{n+1} = x^{n+1}$ and $(x-x^{n+1})^n = 0$, x is the sum of $x-x^{n+1} \in N$ and $x^{n+1} \in E_{n+1}$. Hence $R = N \oplus E_{n+1}$ by Corollary 3, so that $R = E_{n+1}$. The converse is also clear by [4, Theorem 1].

References

[1] H. E. Bell : On commutativity and structure of periodic rings, Math. J. Okayama Univ. **27** (1985), 1–3.

[2] K. Chiba and H. Tominaga : On generalizations of P_1-rings and gsr-rings, Math. J. Okayama Univ. **18** (1976), 149–152.

[3] Y. Hirano : Some studies on strongly π-regular rings, Math. J. Okayama Univ. **20** (1978), 141–149.

[4] H. Komatsu : On the equational definability of addition in rings, Math. J. Okayama Univ. **24** (1982), 133–136.

[5] M. Ōhori : On non-commutative generalized p. p. rings, Math. J. Okayama Univ. **26** (1984), 157–167.

[6] H. Tominaga : Rings decomposed into direct sums of J-rings and nil rings, Internat. J. Math. Math. Sci. 8 (1985), 205–207.

Department of Mathematics
Okayama University

(*Received September 25, 1984*)