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Kazuo Kishimoto*

Hidetoshi Marubayashi[†]

Akira Ueda[‡]

*Shinshu University

[†]Osaka University

[‡]Kobe University

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AN ORE EXTENSION OVER A ν -HC ORDER

Dedicated to Professor Hisao Tominaga on his 60th birthday

KAZUO KISHIMOTO, HIDETOSHI MARUBAYASHI and AKIRA UEDA

The polynomial ring over a hereditary noetherian prime ring (an *HNP* ring for short) is not an *HNP* ring. This leads us to the concept of *VHC* order in a simple artinian ring in [8] (in [4], Fujita defined a ν -*HC* order which is a little more general concept of *VHC* order) and the following problems naturally raise; If R is a ν -*HC* order in a simple artinian ring Q , then so are $R[x; \sigma]$ and $R[x; d]$, where σ is an automorphism of R and d is a derivation of R . In [9], one of the authors proved that $R[x; \sigma]$ is a ν -*HC* order with enough ν -invertible ideals if R is a ν -*HC* order with enough ν -invertible ideals.

In this paper, we define a d - ν -*HC* order with enough d - ν -invertible ideals, and we prove the following theorem; R is a d - ν -*HC* order with enough d - ν -invertible ideals if and only if $R[x; d]$ is a ν -*HC* order with enough ν -invertible ideals (cf. § 1 for the definitions).

The theorem will be proved in § 2 by pointing out all maximal ν -invertible ideals of $R[x; d]$ which are derived from R and $Q[x; d]$.

In § 1, we define the concepts of d - ν -ideals and d - ν -invertible ideals, and give some elementary properties of them, some of which are used to prove the theorem.

In § 3, we briefly discuss on the set of all d - ν -invertible ideals of $R[x; d]$ and the class group of $R[x; d]$ which extend Chamarie's results in the case of a Krull order. Any maximal d - ν -ideal of a d - ν -*HC* order is not necessarily a prime ideal (even a semi prime ideal) as it is seen in Example 1 of § 3, and we give some examples of d - ν -*HC* orders with enough d - ν -invertible ideals.

Concerning terminologies which are not defined in this paper, we refer to [8].

1. Throughout this paper, R will be an order with a derivation d in a simple artinian ring Q . First of all, we recall some notations and definitions in [8] and [4]. Let X and Y be subsets of Q . Then we use the following notations; $(X:Y)_l = \{s \in Q \mid sY \subseteq X\}$ and $(X:Y)_r = \{t \in Q \mid Yt \subseteq X\}$. Let I be a right R -ideal. We define $I_\nu = (R:(R:I)_l)_r$, and if $I = I_\nu$, then it

is called a right ν - R -ideal (a right ν -ideal if there are no confusions). Similarly we define ${}_vJ = (R : (R : J)_r)_l$ for any left R -ideal J and J is called a left ν -ideal if ${}_vJ = J$. An R -ideal A is called a ν -ideal if ${}_vA = A = A_v$. An integral and ν -ideal is simply called a ν -ideal of R . A ν -ideal is called a ν -invertible if $(A(R : A)_r)_v = R = {}_v((R : A)_l)$. Note that a ν -ideal A is ν -invertible if and only if $O_r(A) = R = O_l(A)$ by Lemma 1.1 of [4], where $O_r(A) = \{q \in Q \mid Aq \subseteq A\}$, a right order of A , and $O_l(A) = \{a \in Q \mid qA \subseteq A\}$, a left order of A . If A is ν -invertible, then by Lemma 1.1 of [4], $(R : A)_r = (R : A)_l$ which is denoted by A^{-1} . A ν -ideal A of R is called ν -idempotent if ${}_v(A^2) = A = (A^2)_v$. Let $\mathcal{C} = \mathcal{C}(R)$ be a right Gabriel topology corresponding to the torsion theory cogenerated by $E(Q/R)$ ($E(M)$ denotes the injective hull of a right R -module M). Then $\mathcal{C} = \{C : \text{right ideal of } R \mid (R : r^{-1}C) = R \text{ for any } r \in R\}$, where $r^{-1}C = \{x \in R \mid rx \in C\}$, by Proposition 5.5 of [13, p. 147]. Let I be a right R -ideal and put $cl(I) = \{q \in Q \mid qC \subseteq I \text{ for some } C \in \mathcal{C}\}$. If $I = cl(I)$, then I is called a right closed R -ideal. Similarly we can define the left Gabriel topology $\mathcal{C}'$ on R and a left $\mathcal{C}'$ -closed ideal. Fujita considered the following conditions:

(K) : ${}_v(A(R : A)_l) = O_l(A)$ for any ideal A of R such that $A = {}_vA$ and $((R : B)_r B)_v = O_r(B)$ for any ideal B of R such that $B = B_v$.

(C) : R satisfies the maximum condition on right $\mathcal{C}$ -closed ideals of R as well as left $\mathcal{C}'$ -closed ideals of R .

If an order R satisfies (K) and (C), then it is called a ν -HC order. R is said to have enough ν -invertible ideals if any ν -ideal of R contains a ν -invertible ideal of R . We note that I is closed if $I = I_v$. Hence if R satisfies (K), then R satisfies the maximum condition on one-sided ν -ideals of R . An R -ideal I is called a d -stable ideal (a d -ideal) if $d(I) \subseteq I$. A ν -ideal which is d -stable is called a d - ν -invertible ideal. We consider the following condition:

(K') : ${}_v(A(R : A)_l) = O_l(A)$ for any d -ideal of R such that $A = {}_vA$ and $((R : B)_r B)_v = O_r(B)$ for any d -ideal of R such that $B = B_v$.

If an order R satisfies (K') and (C), then it is called a d - ν -HC order, and R is said to have enough d - ν -invertible ideals if any d - ν -ideal of R contains a d - ν -invertible ideal of R .

We denote by $R[x; d]$ the Ore extension of R in an indeterminate x . Any element in $R[x; d]$ has the following form: $\sum_{i=0}^n r_i x^i$ ($r_i \in R$) with multiplication defined by $xr = rx + d(r)$ for every $r \in R$. A subset S of R is said to be a regular Ore set if any element of S is regular in R and R satisfies the left and right Ore conditions with respect to S . The quotient ring of R with respect to S is denoted by R_s , and d is extended to a derivation of R_s .

in the following way : $d(ac^{-1}) = d(a)c^{-1} - ac^{-1}d(c)c^{-1}$, where $a \in R$ and $c \in S$. As it is easily seen, the following mapping : $q(x) = \sum_{i=0}^n q_i x^i \rightarrow \sum_{i=0}^n d(q_i) x^i$ is a derivation of $Q[x; d]$ which extends the derivation d of Q . We denote it by d again. Then we note that any $R[x; d]$ -ideal I in $Q[x; d]$ is d -stable, because $d(q(x)) = xq(x) - q(x)x \in I$ for any $q(x) \in I$. Let $\mathcal{C}$ be the set of all regular elements of R . Then $\mathcal{C}$ is a regular Ore set of $R[x; d]$ and $R[x; d]_{\mathcal{C}} = Q[x; d]$. Furthermore, $Q[x; d]$ is a prime, left and right principal ideal ring [1]. So it has a classical quotient ring $Q(Q[x; d])$ which is simple artinian. Hence $R[x; d]$ is an order in $Q(Q[x; d])$.

In studying the structure of $R[x; d]$, d -v-ideals and d -v-invertible ideals play an important rôle. So first we shall give some elementary properties of d -v-ideals and d -v-invertible ideals, and the proofs of them are all straightforward ;

Proposition 1.1. (1) *Let I be a right (left) d - R -ideal. Then so is $(R : I)_l ((R : I)_r)$ and thus $I_v(vI)$ is also d -stable.*

(2) *Let A be a d - R -ideal. Then $AR[x; d]$ is an $R[x; d]$ -ideal denoted by $A[x; d]$, and $O_l(A)$ and $O_r(A)$ are also d -stable.*

(3) *Let A and B be d - R -ideals. Then so is AB and $ABR[x; d] = A[x; d]B[x; d]$.*

(4) *Let I be a right R -ideal and J a left R -ideal. Then $(R[x; d] : IR[x; d])_l = R[x; d](R : I)_l$ and $(R[x; d] : R[x; d]J)_r = (R : J)_r R[x; d]$. In particular, $(IR[x; d])_v = I_v R[x; d]$ and $_v(R[x; d]J) = R[x; d]_v J$.*

(5) *Let A be a d -v-ideal. Then so is $A[x; d]$.*

(6) *Let A be a d -v-invertible ideal. Then so is $A[x; d]$ and $(A[x; d])^{-1} = A^{-1}[x; d]$.*

(7) *Let A be a d -stable and v-idempotent ideal of R (a d -v-idempotent ideal). Then so is $A[x; d]$.*

Lemma 1.2. *Let R be a d -v-HC order.*

(1) *If A is a d -ideal of R , then $A_v = _v A$.*

(2) *If A is a d - R -ideal and there is a d -v-invertible ideal B such that $B^{-1} \supseteq A$, then $A_v = _v A$.*

Proof. As in Lemma 1.5 and Corollary 1.6 of [4].

A d -v-ideal M of R is called maximal if it is maximal amongst all d -v-ideals of R . As it is easily seen from an example at the end of this paper,

we can't expect that any maximal d - v -ideal is a prime ideal (even a semi-prime ideal). But we have the following nice property ;

Lemma 1.3. *Let R be an order with a derivation d and let M be a d -prime ideal of R , i. e., $AB \subseteq M$ implies either $A \subseteq M$ or $B \subseteq M$, where A and B are d -ideals of R , then $M[x; d]$ is a prime ideal.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be ideals of $R[x; d]$ such that $\mathfrak{a}\mathfrak{b} \subseteq M[x; d]$, and assume that $\mathfrak{b} \not\subseteq M[x; d]$. Put $C(\mathfrak{b}) = \{b_n \mid b_n x^n + \cdots + b_0 \in \mathfrak{b}\} \cup \{0\}$. Then $C(\mathfrak{b})$ is a d -ideal of R and $C(\mathfrak{b}) \not\subseteq M$. Let $a(x) = a_m x^m + \cdots + a_0 \in \mathfrak{a}$. For any $b_n \in C(\mathfrak{b})$, there is $b(x) \in \mathfrak{b}$ such that $b(x) = b_n x^n + \cdots + b_0 \in \mathfrak{b}$. Since $a(x)b(x) \in M[x; d]$, it follows that $a_m b_n \in M$. So $a_m C(\mathfrak{b}) \subseteq M$ and $a_m \in M$. Repeating this process, we obtain $a(x) \in M[x; d]$, and so $\mathfrak{a} \subseteq M[x; d]$, proving the lemma.

As a corollary to Lemma 1.3, we have

Corollary 1.4. *Let R be a d - v -HC order and let M be a maximal d - v -ideal of R , then M is a d -prime ideal and so $M[x; d]$ is a prime ideal of $R[x; d]$.*

Proof. Let A and B be d -ideals of R and $AB \subseteq M$. Assume that $B \not\subseteq M$, then $B_v \not\subseteq M$ and B_v is d -stable. So $B_v = R$. Hence $A = Ar = AB_v \subseteq (AB_v)_v = (AB)_v = M$ by lemma 1.1 of [7]. It follows that M is d -prime.

Lemma 1.5. *Let R be a d - v -HC order. Then a maximal d - v -ideal of R is either v -idempotent or v -invertible.*

Proof. As in Lemma 1.5 of [7].

A finite set of distinct maximal d -ideals $M_1, \dots, M_n$ of R which are v -idempotent is called a d - v -cycle if $O_r(M_1) = O_l(M_2), \dots, O_r(M_n) = O_l(M_1)$. A maximal d - v -ideal M which is v -invertible is also considered as a d - v -cycle, because $O_r(M) = O_l(M)$.

The following proposition will be proved by combining the methods in [7] with Corollary 1.4, and these will be used in § 2 to study the structure of $R[x; d]$.

Proposition 1.6. *Let R be a d - v -HC order. Then*

(1) *Let M_1 be a maximal d - v -ideal such that M_1 is v -idempotent and it*

contains a d - v -invertible ideal X of R . Then there exist maximal d - v -ideals $M_2, \dots, M_n$ such that $M_i \supseteq X$, M_i is v -idempotent and $M_1, \dots, M_n$ is a d - v -cycle.

(2) Let P be an ideal of R . Then P is a maximal d - v -invertible ideal of R (maximal amongst all d - v -invertible ideals of R) if and only if it is an intersection of a d - v -cycle.

(3) If R has enough d - v -invertible ideals, then the set $D_d(R)$ of all d - v -invertible ideals is a free abelian group generated by maximal d - v -invertible ideals.

2. In this section, we shall prove the main theorem mentioned in the introduction. Let us start off the following lemma whose proof is similar to one of Theorem 3.1.8 of [3].

Lemma 2.1. *Let R be an order in Q . Then $R[x; d]$ satisfies the condition (C) if R satisfies the condition (C).*

Let R be an order in Q satisfying the maximum condition on one-sided v -ideals of R and let $\mathcal{P}$ be a set consisting of v -invertible ideals of R which is closed by " v -multiplication", i.e., if X and $Y \in \mathcal{P}$, then $(XY)_v \in \mathcal{P}$. Then $T = \bigcup X^{-1} (X \in \mathcal{P})$ is an overring of R and we have the following :

Lemma 2.2. (1) *If $X \in \mathcal{P}$, then $X \in \mathcal{G}$, where $\mathcal{G} = \{I : \text{right ideal of } R \mid \text{Hom}_R(R/I, E(Q/T)) = 0\}$.*

(2) *$T = R_s$, where R_s denotes the ring of right quotients of R with respect to $\mathcal{G}$.*

(3) *For any $I \in \mathcal{G}$, there exists $X \in \mathcal{P}$ such that $I_v \supseteq X$.*

Proof. (1) Let X be a d - v -invertible ideal of R . If $\text{Hom}(R/X, E(Q/T)) \neq 0$, then there is a non-zero f in it, in particular, $f(\bar{1}) \neq 0$, where $\bar{1} = [1 + X]$ in R/X . Since Q/T is essential in $E(Q/T)$, there is a non zero element r in R such that $f(\bar{1}) \in Q/T$ and $f(\bar{1})r \neq 0$. On the other hand, $\bar{0} = f(\bar{X}) \supseteq f(\bar{1})rX = [q + T]X$, where q is in Q but not in T . Hence $qX \subseteq T = \bigcup Y^{-1}$ and $qX \subseteq Y^{-1}$ for some $Y \in \mathcal{P}$, because X is finitely generated as a right v -ideal. It follows that $q \in (Y^{-1}X^{-1})_v = (XY)^{-1} \subseteq T$. This is a contradiction.

(2) From (1), $R_s \supseteq T$. So $R_s/T \subseteq Q/T \subseteq E(Q/T)$. This implies that R_s/T is $\mathcal{G}$ -torsion free. On the other hand, $R_s/R \rightarrow R_s/T \rightarrow 0$ is exact and R_s/R is $\mathcal{G}$ -torsion. Hence R_s/T is $\mathcal{G}$ -torsion. It follows that $R_s/T = 0$, i.e., $R_s = T$.

(3) For $I \in \mathcal{G}$, we have $(R:I)_t \subseteq R_s = T$. Since $(R:I)_t$ is finitely generated as a left ν -ideal, $(R:I)_t X \subseteq R$ for some $X \in \mathcal{P}$. This implies $X \subseteq I_\nu$.

Remark. Put $\mathcal{G}' = \{J; \text{left ideal of } R \mid \text{Hom}_R(R/J, E'(Q/T)) = 0\}$ ($E'(N)$ denotes the injective hull of a left R -module N). Then, by the left version of Lemma 2.2, we have $T = {}_{\mathcal{G}'}R$ (${}_{\mathcal{G}'}R$ denotes the ring of quotients of R with respect to $\mathcal{G}'$).

Lemma 2.3. *Under the same assumption as in Lemma 2.2, let I be a right R -ideal. Then*

- (1) $(IT)_\nu = (I_\nu T)_\nu$.
- (2) Let J be an R -ideal. Then $(IJT)_\nu = (I(JT)_\nu)_\nu$.

Proof. (1) It is clear that $(IT)_\nu \subseteq (I_\nu T)_\nu$. To prove the converse inclusion, let c be a unit in Q such that $IT \subseteq cT$. Then $c^{-1}I \subseteq T$. Since $c^{-1}I$ is finitely generated as a right ν -ideal, there exists $X \in \mathcal{P}$ such that $c^{-1}I \subseteq (c^{-1}I)_\nu = c^{-1}I_\nu \subseteq X^{-1}$. Hence $I_\nu \subseteq cX^{-1} \subseteq cT$ and so $I_\nu T \subseteq cT$. This implies that $(I_\nu T)_\nu \subseteq (IT)_\nu$ by Proposition 4.1 of [7] and thus $(IT)_\nu = (I_\nu T)_\nu$.

(2) is proved by the same way as in (1).

For a d - ν -HC order R , we put $R_d = \bigcup X^{-1}$, where X runs over all d - ν -invertible ideals of R . Then R_d is an overring of R and $d(R_d) \subseteq R_d$. We have

Lemma 2.4. *Let R be a d - ν -HC order. Then*

- (1) R_d is a d - ν -HC order.
- (2) Assume that T has enough d - ν -invertible ideals. Then R_d has no proper d - ν -ideals of R_d and $R_d[x; d]$ is a Krull order in the sense of [3].

Proof. (1) Because the set of all d - ν -invertible ideals of R is closed by the ν -multiplication, R_d satisfies (C) by Lemma 2.4 of [8] and Lemma 2.2. To prove that R_d satisfies (K'), we adopt the method used in Proposition 4.1 of [4]. Let A' be any d -ideal of R_d such that $A' = {}_\nu A'$. We put $A = A' \cap R$. Then by Lemma 2.2 and Lemma 2.3 of [8], ${}_s A = \{q \in Q \mid Jq \subseteq A \text{ for some } J \in \mathcal{F}'\} = A' \subseteq A_s = \{q \in Q \mid qI \subseteq A \text{ for some } I \in \mathcal{F}\}$, where $\mathcal{F} = \{I; \text{right ideal} \mid \text{Hom}_R(R/I, E(Q/R_d)) = 0\}$ and $\mathcal{F}' = \{J; \text{left ideal} \mid \text{Hom}_R(R/J, E'(Q/R_d)) = 0\}$, ($E'(Q/R_d)$ denotes the left R -injective hull

of Q/R_d). Hence ${}_v(A'(R_d : A)_l) \supseteq {}_v({}_s A(R_d : A_F)_l) \supseteq {}_v(R_d A(R : A)_l) = {}_s({}_v(A(R : A)_l)) = {}_s O_l(A)$. Since ${}_v(A'(R_d : A)_l)$ is a right $O_l(A)$ -module and $1 \in O_l(A)$, we have ${}_v(A'(R_d : A)_l) = O_l(A')$. It is proved by the similar way that $((R_d : B')_r B')_v = O_r(B')$ for any d -ideal B' of R_d such that $B' = B'_v$. Hence R_d is a d -v-HC order.

(2) Let A' be any d -ideal of R_d such that $A' = A'_v$. Then $A = A' = \cap R$ is a d -v-ideal of R by Lemma 2.3 of [8]. So there exists a d -v-invertible ideal X of R contained in A . Hence we have $R_d \supseteq A' = A_s \supseteq X_s = R_d$ by Lemma 2.3 of [8] and Lemma 2.2., and so $A' = R_d$. Hence R_d has no proper d -v-ideals. To prove that $R_d[x; d]$ is a Krull order, let B be any d -ideal of R_d and let q be any element in $O_l(B)$. Then $qB \subseteq B$ implies that $qB_v \subseteq B_v$. But $B_v = R_d$, because B_v is a d -v-ideal, and so $q \in R_d$. Hence $O_l(B) = R_d$ and similarly $O_r(B) = R_d$. So $R_d[x; d]$ is a maximal order by Proposition 3.1.4 of [3].

Let R be an order in Q . We denote by $S(R)$, the Asano overring of R , i. e., $S(R) = \cup X^{-1}$ (X runs over all v -invertible ideals of R). Let A be an ideal of R . If $C(A) = \{c \in R \mid c \text{ is regular mod } A\}$ is a regular Ore set, then we denote $R_{C(A)}$ by R_A .

Lemma 2.5. *Let R be a d -v-HC order and let B be any maximal d -v-invertible ideal of R . Then*

(1) *$B[x; d]$ is a semi-prime ideal of $R[x; d]$ and intersection of a cycle in the sense of [9].*

(2) *$R[x; d]_{B[x; d]}$ exists and is an HNP ring whose Jacobson radical $B[x; d]R[x; d]_{B[x; d]}$ is a unique maximal invertible ideal.*

Proof. (1) By Proposition 1.6, $B = M_1 \cap \dots \cap M_n$, where $M_1, \dots, M_n$ is a d -v-cycle. Hence $B[x; d] = M_1[x; d] \cap \dots \cap M_n[x; d]$ is a semi-prime ideal by Corollary 1.4. Furthermore, since $O_r(M_i[x; d]) = O_r(M_i)[x; d] = O_l(M_{i+1})[x; d] = O_l(M_{i+1}[x; d])$ and $((M_i[x; d]^2)_v = (M_i^2[x; d])_v = (M_i^2)_v[x; d] = M_i[x; d]$. Thus $M_1[x; d], \dots, M_n[x; d]$ is a cycle in the sense of [9].

(2) This follows from Lemma 2.1 of [9], Lemma 2.1 and (1).

Lemma 2.6. *Let R be a d -v-HC order with enough d -v-invertible ideals. Then*

(1) *$R[x; d] = (\cap R[x; d]_{B[x; d]}) \cap R_d[x; d]$, where B runs over all d -v-invertible ideals of R .*

$$(2) \quad R_d[x; d] = S(R[x; d]) \cap Q[x; d].$$

Proof. The lemma is proved by the exact same way as in Lemma 2.9 of [9].

Lemma 2.7. *Let R be a d - v -HC order with enough d - v -invertible ideals and let A' be any non-zero ideal of $Q[x; d]$. Then*

(1) $A = A' \cap R[x; d]$ is a v -invertible ideal of $R[x; d]$.

(2) $A = (A_1^{n_1} \dots A_k^{n_k}) = (A_1^{n_1})_v \cap \dots \cap (A_k^{n_k})_v$ for some $n_i > 0$, where A_i is a maximal v -ideal of $R[x; d]$ which is v -invertible.

Proof. As in Lemma 2.12 of [9].

Theorem 2.8. *Let R be an order with a derivation d in a simple artinian ring Q . Then R is a d - v -HC order with enough d - v -invertible ideals if and only if $R[x; d]$ is a v -HC order with enough v -invertible ideals.*

Proof. To prove the necessity, let R be a d - v -HC order with enough d - v -invertible ideals. Then $R[x; d]$ satisfies (C) by Lemma 2.1. Next we shall prove that $R[x; d]$ has enough v -invertible ideals. To do this let A be any ideal of $R[x; d]$ such that $A_v = A$. (i) In the case $A \cap R \neq 0$, $(A \cap R)_v$ is a v -ideal by Lemma 1.2, because A is a d -ideal. Hence there is a d - v -invertible ideal X of R such that $(A \cap R)_v \supseteq X$. Then $X[x; d]$ is a v -invertible ideal of $R[x; d]$ which is contained in A . (ii) In the case $A \cap R = 0$. Because $Q[x; d] = R[x; d]_{\mathcal{C}}$ is hereditary, where $\mathcal{C}$ is the set of all regular elements in R , $AQ[x; d]$ is an ideal of $Q[x; d]$ by Lemma 2.3 of [4]. So $B = AQ[x; d] \cap R[x; d]$ is a v -invertible ideal of $R[x; d]$ by Lemma 2.7. If $A = B$, then there is nothing to state any more. Assume that $B \subsetneq A$. Then $C = \{r \in R \mid rB \subseteq A\}$ is a non-zero and d -ideal of R , because $BQ[x; d] = AQ[x; d]$, A and B are both d -stable, and B is finitely generated as a right v -ideal. Hence $C[x; d]$ is an ideal of $R[x; d]$ such that $C[x; d]B \subseteq A$. So $C = C_v$ by Lemma 1.1 of [8], because B is v -invertible. Since C is a v -ideal by Lemma 1.2, there is a d - v -invertible ideal D of R such that $D \subseteq C$. Then $(D[x; d]B)_v \subseteq A_v = A$ and $(D[x; d]B)_v$ is a d - v -invertible ideal of $R[x; d]$. Thus every ideal A of $R[x; d]$ such that $A_v = A$ contains a v -invertible ideal of $R[x; d]$. In particular, $R[x; d]$ has enough v -invertible ideals. To prove that $R[x; d]$ satisfies (K), let C be any ideal of $R[x; d]$ such that $C = C_v$. By Lemma 2.6, we have $R[x; d] = (\cap R[x; d]_{B[x; d]}) \cap Q[x; d] \cap S(R[x; d])$ and so $I_v = (\cap IR[x; d]_{B[x; d]}) \cap$

$IQ[x; d] \cap (I_v S(R[x; d]))_v$ by the same way as in Lemma 2.7 of [2], where $I = (R[x; d] : C)_r C$. Since $R[x; d]_{B[x; d]}$ and $Q[x; d]$ are hereditary, we have $IR[x; d]_{B[x; d]} = (R[x; d]_{B[x; d]} : CR[x; d]_{B[x; d]})_r CR[x; d]_{B[x; d]} = O_r(CR[x; d])$ by using Lemma 2.3 of [8] (also see Lemma 2.3 of [4]). Similarly, $IQ[x; d] = O_r(CQ[x; d])$. To prove that $(I_v S(R[x; d]))_v = S(R[x; d])$, let Y be a v -invertible ideal of $R[x; d]$ which is contained in C (the existence of Y is guaranteed by the proof above). So it follows that $S(R[x; d]) \supseteq (CS(R[x; d]))_v \supseteq (YS(R[x; d]))_v = S(R[x; d])$ (the last equality follows from Lemma 2.3 of [8] and Lemma 2.2), and $S(R[x; d]) \subseteq (R[x; d] : C)_r S(R[x; d]) \subseteq Y^{-1} S(R[x; d]) \subseteq S(R[x; d])$. Hence $((R[x; d] : C)_r C)_v S(R[x; d])_v = ((R[x; d] : C)_r (CS(R[x; d]))_v)_v = S(R[x; d])$ by Lemma 2.3. Thus we have $I_v \ni 1$ and so $I_v = O_r(C)$ as desired. Similarly, we can prove $O_l(D) = {}_v(D(R[x; d] : D))_l$ for any ideal D of R with $D = {}_v D$, finishing the proof of the necessity. The sufficiency of the theorem follows from the following Lemmas 2.9, 2.10 and 2.13.

Lemma 2.9. *Let $R[x; d]$ be a v -HC order with enough v -invertible ideals. Then R satisfies (K') .*

Proof. Let A be a d -ideal of R such that ${}_v A = A$. Then $A[x; d]$ is an ideal of $R[x; d]$ which is a v -ideal. So $R[x; d]_v(A(R : A))_l = {}_v(A[x; d] \cdot (R[x; d] : A[x; d]))_l = O_l(A[x; d]) = O_l(A)[x; d]$. Hence ${}_v(A(R : A))_l = O_l(A)$. It is proved similarly that $((R : B)_r B)_v = O_r(B)$ for any d -ideal B of R such that $B_v = B$. Hence R satisfies (K') .

Lemma 2.10. *Let $R[x; d]$ be a v -HC order with enough v -invertible ideals. Then R satisfies (C) .*

Proof. Let E and E^* be the injective hulls of Q/R and $Q(R[x; d]/R[x; d])$ as a right R -module and a right $R[x; d]$ -module respectively. First we show that E^* is injective as a right R -module. Let A be any right ideal of R and let f be any R -homomorphism from A to E^* . Then $AR[x; d]$ is a right ideal of $R[x; d]$ and the mapping $f' : AR[x; d] \rightarrow E^*$ given by $f'(a_m x^m + \cdots + a_0) = f(a_m)x^m + \cdots + f(a_0)$ is an $R[x; d]$ -homomorphism, where $a_i \in A$. Hence there is an element t in E^* such that $f'(a(x)) = ta(x)$ for any $a(x) \in AR[x; d]$. Then $f(a) = ta$ for any $a \in A$. So E^* is injective as a right R -module. It follows that $E \subseteq E^*$, because $Q/R \subseteq Q(R[x; d])/R[x; d] \subseteq E^*$. So R satisfies (C) by Proposition 2.4 of [13, p. 264].

Lemma 2.11. *Let $R[x; d]$ be a ν -HC order with enough ν -invertible ideals, and let M be a maximal ν -ideal of $R[x; d]$. Then $M = A = A' \cap R[x; d]$ for some maximal ideal A' of $Q[x; d]$, or $M = (M \cap R)[x; d]$ and $M \cap R$ is a maximal d - ν -ideal if $M \cap R \neq 0$. Furthermore, $A = A' \cap R[x; d]$ (A' is a maximal ideal of $Q[x; d]$) is a maximal ν -ideal and ν -invertible.*

Proof. (i) In the case $M \cap R = 0$. There is a maximal ideal A' of $Q[x; d]$ such that $MQ[x; d] \subseteq A' \subseteq Q[x; d]$. So $M \subseteq MQ[x; d] \cap R \subseteq A \subseteq R[x; d]$ and A is a ν -ideal. Hence $M = A$. (ii) In the case $M \cap R \neq 0$. $M \cap R$ is a d -prime ideal of R . So $(M \cap R)[x; d]$ is a prime ideal of $R[x; d]$ by Lemma 1.3. Hence $(M \cap R)[x; d]$ is a maximal ν -ideal by Lemma 1.2 of [10], and so $M \cap R$ is a maximal d - ν -ideal of R . Finally for any maximal ideal A' of $Q[x; d]$, $A = A' \cap R[x; d]$ is clearly a prime ν -ideal. So it is a maximal ν -ideal by Lemma 1.2 of [10]. It follows that either A is ν -idempotent or ν -invertible by Lemma 1.5 of [8]. Assume that A is ν -idempotent. Then $AQ[x; d]$ is also ν -idempotent. This is a contradiction. Hence A is ν -invertible.

Lemma 2.12. *Let $R[x; d]$ be a ν -HC order with enough ν -invertible ideals and let P be a maximal ν -invertible ideal of $R[x; d]$. Then either $P = A = A' \cap R[x; d]$ or $P = M_1[x; d] \cap \dots \cap M_n[x; d]$, where A' is a maximal ideal of $Q[x; d]$, $M_1, \dots, M_n$ is a d - ν -cycle of R and $P_o = M_1 \cap \dots \cap M_n$ is a maximal d - ν -invertible ideal of $R[x; d]$.*

Proof. (i) In the case $P \cap R = 0$. There is a maximal ideal A' of $Q[x; d]$ such that $PQ[x; d] \subseteq A' \subseteq Q[x; d]$. So $P \subseteq A$. This implies that $P = A$. (ii) In the case $P \cap R \neq 0$. Since it is a d - ν -ideal of R , $(P \cap R)[x; d]$ is a ν -ideal of $R[x; d]$, and by Lemma 1.12 of [8], there is a cycle $N_1, \dots, N_n$. Since N_i is a maximal ν -ideal, $N_i = A$ or $N_i = M_i[x; d]$, where $M_i = N_i \cap R$ by Lemma 2.11. If $N_i = A$, then $Q[x; d] = (P \cap R) \cdot Q[x; d] \subseteq N_i Q[x; d] = A'$ by Lemma 2.3 of [8], a contradiction. So $N_i = M_i[x; d]$, and $M_1, \dots, M_n$ is a d -cycle of R by Proposition 1.1. Hence $P = M_1[x; d] \cap \dots \cap M_n[x; d] = P_o[x; d]$, $P_o = M_1 \cap \dots \cap M_n$ is a maximal d - ν -invertible ideal by Proposition 1.6.

Lemma 2.13. *Let $R[x; d]$ be a ν -HC order with enough ν -invertible ideals. Then R has enough d - ν -invertible ideals.*

Proof. Let V be any d - ν -ideal of R . Then by Theorem 2.23 of [8] and

Lemma 2.11, we have $R[x; d] = (\cap R[x; d]_P) \cap (\cap R[x; d]_A) \cap S(R[x; d])$, where P and A are as in Lemma 2.11. Since $R[x; d]_A = Q[x; d]_A$, by Proposition 1.1 of [7], it follows that $VR[x; d]_A = R[x; d]_A$. Furthermore, $(VS(R[x; d]))_v = S(R[x; d])$, because $V[x; d]$ contains a v -invertible ideal of $R[x; d]$. Thus we have $V[x; d] = (\cap V[x; d]R[x; d]_P) \cap (\cap R[x; d]_A) \cap S(R[x; d])$. So there are finitely many maximal v -invertible ideals $P_1, \dots, P_k$ of $R[x; d]$ such that $R[x; d]_{P_i} \cong V[x; d]R[x; d]_{P_i}$. Since each $R[x; d]_{P_i}$ is an HNP ring whose Jacobson radical $P_i' = P_iR[x; d]_{P_i}$ is a unique maximal invertible ideal, we have $V[x; d]R[x; d]_{P_i} \cong P_i'^{n_i}$ for some $n_i > 0$. Thus it follows that $V[x; d] \cong (P_1R[x; d]_{P_1})^{n_1} \cap \dots \cap (P_kR[x; d]_{P_k})^{n_k} \cap R[x; d] \cong (P_1^{n_1})_v \cap \dots \cap (P_k^{n_k})_v = (P_1^{n_1} \dots P_k^{n_k})_v = (P_{10}^{n_1} \dots P_{k0}^{n_k}[x; d])_v = (P_{10}^{n_1} \dots P_{k0}^{n_k})_v[x; d]$. Hence $V \cong (P_{10}^{n_1} \dots P_{k0}^{n_k})_v$, and $(P_{10}^{n_1} \dots P_{k0}^{n_k})_v$ is a d - v -invertible ideal of R .

Corollary 2.14. *Let R be a d - v -HC order with enough d - v -invertible ideals. Then $R[x; d] = (\cap R[x; d]_{B[x; d]}) \cap (\cap R[x; d]_A) \cap S(R[x; d])$, where B runs over all maximal d - v -invertible ideals of R , $A = A' \cap R[x; d]$, and A' runs over all maximal ideal of $Q[x; d]$. Furthermore, $R[x; d]_{B[x; d]}$ is an HNP ring whose Jacobson radical is a unique maximal invertible ideal, $R[x; d]_A$ is a local, Dedekind prime ring and $S(R[x; d])$ is a v -simple (i. e., it has no proper v -ideal of $S(R[x; d])$) and a Krull order.*

Proof. Since $Q[x; d]$ is a Dedekind prime ring, we have $Q[x; d] = (\cap Q[x; d]_{A'}) \cap S(R[x; d])$, where A' runs over all maximal ideals of $Q[x; d]$ and $Q[x; d]_{A'}$ is a local Dedekind prime ring by Theorem 3.1 of [5]. Put $A = A' \cap R[x; d]$. Then $R[x; d]_A = Q[x; d]_{A'}$. Hence the assertion follows from Theorem 2.23 of [8] and Theorem 2.8 (note that $S(R[x; d]) \cong S(Q[x; d])$), see the proof of Lemma 2.15 of [9].

3. Let R be an order in Q and let A be any R -ideal such that $aR = A = Ra'$ for some $a, a' \in Q$. Then $Ra = A = aR$ by the same way as in [6, p. 37]. We denote by $D(R)$ the group consisting of all v -invertible ideals in which the multiplication is given by $X \circ Y = (XY)_v$ for any $X, Y \in D(R)$ and denote by $P(R)$ the subgroup of $D(R)$ consisting of all principal R -ideals. Put $C(R) = D(R)/P(R)$, called the class group of R . Similarly we can define $C_d(R) = D_d(R)/P_d(R)$, the d -class group of R , where $P_d(R) = D_d(R) \cap P(R)$. Now let R be a d - v -HC order with enough d - v -invertible ideals, then we have the following :

(1) The set $\{B[x; d], A \mid B: \text{maximal } d\text{-}\nu\text{-invertible ideal of } R \text{ and } A = A' \cap R[x; d], \text{ where } A' \text{ is a maximal ideal of } Q[x; d]\}$ is the full set of maximal ν -invertible ideals of $R[x; d]$ (see the proof of Corollary 4.13 of [11]).

(2) Let I be a ν -invertible ideal such that $Q[x; d] \cong I$ and $I \cap Q \neq 0$. Then $I \cap Q$ is ν -invertible and $I = (I \cap Q)[x; d]$ (see the proof of Lemma 2.18 of [9]).

In case R is a Krull order, Chamarie has proved the property (2) above by using a complex lemma (see Lemma 3.3.1 of [3]) and he has obtained the following proposition in case R is a Krull order.

Proposition 3.1. ([3, Theorem 3.3.5]) *Let R be a $d\text{-}\nu\text{-HC}$ order with enough $d\text{-}\nu\text{-invertible}$ ideals. Then*

(1) $D(R[x; d]) \cong D_a(R) \oplus D(Q[x; d])$.

(2) *The mapping $f: D_a(R) \rightarrow D(R[x; d])$ given by $f(a) = a[x; d]$, where $a \in D_a(R)$, induces the epimorphism $\bar{f}: C_a(R) \rightarrow C(R[x; d])$ and if R is a domain, then $\bar{f}$ is an isomorphism.*

Proof. (1) follows from Theorem 1.13 of [8], Proposition 1.6, Theorem 2.8 and property (1) above. (2) follows from the proof of Theorem 3.3.5 of [3].

We shall end this paper with several examples. We have pointed out in the paragraph before Lemma 1.3 that a maximal $d\text{-}\nu$ -ideal is not necessary to be a prime ideal as it is seen in the following example (even it is not a semi-prime ideal). Note that any maximal ν -ideal is a prime ideal (see Lemma 1.4 of [8]). Let σ be any automorphism of a $\nu\text{-HC}$ order R . Then we note that any maximal σ -invariant, ν -ideal is also a prime ideal. Furthermore, any maximal σ -invariant, ν -invertible ideal is a semi-prime ideal. But in case of a derivation type, a maximal $d\text{-}\nu$ -invertible ideal is not necessary to be a semi-prime ideal. This is also seen in Example 1.

Example 1. Let k be a field of $\text{char}(k) = p \neq 0$. Put $R = K[x_1, x_2]$, the polynomial ring over k in two indeterminates x_1 and x_2 . Then R is a commutative Krull ring. Let $d = x_2 \partial_1 + x_1 \partial_2$ be a derivation of R , where $\partial_i x_j = \delta_{ij}$. Put $A_1 = (x_1)$ and $A_2 = (x_2)$, the principal ideals generated by x_1 and x_2 respectively. Then A_1 and A_2 are maximal ν -ideals (of course, these are maximal ν -invertible ideals of R), but not d -stable. On the other hand, $A_1^p \cap A_2^p$ is clearly a maximal $d\text{-}\nu$ -ideal of R and also a maximal $d\text{-}\nu$ -invertible

ideal of R , because R is a Krull order. Obviously $A_1^p \cap A_2^p$ is not semi-prime.

Let R be a d - v -HC order with enough d - v -invertible ideals. Then Example 1 also shows that all generators of $D_d(R)$ are not necessary to be semi-prime, though $D_d(R)$ is a free abelian group.

Example 2. Let R be Krull order in the sense of [3] and let d be any derivation of R . Then R is a d - v -HC order with enough d - v -invertible ideals.

Example 3. Let R be a v -HC order with enough v -invertible ideals. Put $R = T[x]$ and let d be the usual derivation of R . Then R is a d - v -HC order with enough d - v -invertible ideals. In particular, if T is not Krull, then R is not Krull, either. Furthermore, the following result hold :

- (1) $D_d(R) \cong D(T) \oplus D_d(Q[x])$, where Q is a quotient ring of T .
- (2) If $\text{Char}(T) = 0$, then $D_d(Q[x]) = 0$.
- (3) If $\text{Char}(T) = p \neq 0$, then $D_d(Q[x]) \neq 0$.

Proof. By Theorem 2.16 of [9], $R = T[x]$ is a v -HC order and hence it is clearly a d - v -HC order. It is proved by similar way as in the proof of Theorem 2.8 that R has enough d - v -invertible ideals. Next assume that T is not Krull. Then there is an ideal A of T such that either $O_l(A) \cong T$ or $O_r(A) \cong T$ by Proposition 3.1 of [12, p.7]. If $O_l(A) \cong T$, then $O_l(A[x]) = O_l(A)[x] \cong T[x]$. This shows that R is not maximal. (1) follows from Theorem 2.19 of [9] and Proposition 1.6. (2) Since any ideal of $Q[x]$ is principal, it is not d -stable if $\text{Char}(T) = 0$. Hence $D_d(Q[x]) = 0$. (3) The ideal (x^p) , generated by x^p , is clearly d -stable. So $D_d(Q[x]) \neq 0$.

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DEPARTMENT OF MATHEMATICS

FACULTY OF SCIENCE

SHINSHU UNIVERSITY

DEPARTMENT OF MATHEMATICS

COLLEGE OF GENERAL EDUCATION

OSAKA UNIVERSITY

GRADUATE SCHOOL OF SCIENCE AND TECHNOLOGY

KOBE UNIVERSITY

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