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ON STRONGLY π -REGULAR RINGS AND PERIODIC RINGS

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Throughout, R will represent a ring with Jacobson radical J . Let $N = N(R)$ be the set of nilpotent elements in R . Given an integer $q > 1$, we put $E_q = \{u \in R \mid u^q = 0\}$; in particular $E = E_2$. We denote by E_∞ the set of potent elements in R : $E_\infty = \bigcup_{q>1} E_q$. Now, let x be an element of R .

First, x is said to be *regular* if $xyx = x$ for some $y \in R$, while x is said to be *right* (resp. *left*) *regular* if $x^2y = x$ (resp. $yx^2 = x$) for some $y \in R$: x is *strongly regular* if it is both right and left regular. We denote by S the set of strongly regular elements in R . Next, x is called a *right p. p. element* if there exists an $e \in E$ such that $xe = x$ and $r(x) = r(e)$, where $r(*)$ denotes the right annihilator of $*$ in R , and we denote by P the set of right p. p. elements in R . If every element in E is central, R is said to be *normal*. A ring R is called *periodic* if for every $x \in R$ there exist distinct positive integers h, k such that $x^h = x^k$, or equivalently, $x^n \in E_\infty$ for some positive integer n ; R is called *π -regular* (resp. *strongly π -regular*) if there exists a positive integer n such that x^n is regular (resp. strongly regular). Finally, R is called a *generalized right p. p. ring* if for every $x \in R$ there exists a positive integer n such that $x^n \in P$. Obviously, every periodic ring is strongly π -regular, and every π -regular ring is a generalized right p. p. ring (see [5]).

The present objective is to prove the following theorems.

Theorem 1. *If R is normal, then the following are equivalent :*

- 1) *R is a generalized right p. p. ring.*
- 2) *Every $x \in R$ can be written in the form $x = u + a$, where $u \in P$, $a \in N$ and $ua = au$.*

Theorem 2. (1) *The following are equivalent :*

- 1) *R is strongly π -regular.*
- 2) *Every $x \in R$ can be written in the form $x = u + a$, where $u \in S$, $a \in N$ and $ua = au$.*

(2) *Let R be a normal (strongly) π -regular ring. Then N forms an ideal, R/N is strongly regular, and R is a subdirect sum of nil rings and/or local rings with radical nil.*

Theorem 3. (1) *The following are equivalent :*

- 1) *R is periodic.*
- 2) *For every $x \in R$ there exists a positive integer n and an $x' \in \langle x \rangle$ such that $x^n = x^{n+1}x'$.*
- 3) *Every $x \in R$ can be written in the form $x = u + a$, where $u \in E_\infty$, $a \in N$ and $ua = au$.*

(2) *Let R be a normal periodic ring. Then N forms an ideal, R/N is a subdirect sum of periodic fields, and R is a subdirect sum of nil rings and/or local rings R_i such that $R_i/N(R_i)$ is a periodic field.*

Proof of Theorem 1. 1) $\Rightarrow$ 2). Given $x \in R$, we can find a positive integer m and an $e \in E$ such that $x^m e = x^m$ and $r(x^m) = r(e)$. Clearly, $(xe)e = xe$ and $r(xe) = r(e)$, and so $xe \in P$. Since e is central, we have $(x - xe)^m = 0$. Therefore x is the sum of $xe \in P$ and $x - xe \in N$.

2) $\Rightarrow$ 1). Let x be an arbitrary element of R . Then $x = u + a$, where $ue = u$ and $r(u) = r(e)$ for some (central) idempotent e , $a^m = 0$ for some positive integer m and $ua = au$. Since $(x - u)^m = 0$, we can easily see that $x^m e = x^m$. Furthermore, $r(x) \subseteq r(u^m) \subseteq r(e)$. Hence $x^m y = 0$ implies $x^{m-1}ey = ex^{m-1}y = 0$, and therefore $x^{m-2}e^2y = 0$; eventually, $ey = 0$. This proves that $r(x^m) = r(e)$ and $x^m \in P$.

In advance of proving Theorem 2, we state the following easy lemma.

Lemma 1. *Let R be a normal (strongly) π -regular ring. If R is indecomposable, then R is either a nil ring or a local ring with radical N .*

Proof. Suppose $R \neq N$. Then R has a unity and every element of R is either nilpotent or invertible. Hence R is a local ring with radical N .

Proof of Theorem 2. (1) 1) $\Rightarrow$ 2). Let x be an arbitrary element of R . Then, by [1, Lemma 1], there exists a positive integer n and $y \in R$ such that $x^{2n}y = x^n$ and $xy = yx$. It is easy to see that $(x^{n+1}y)^2 x^{n-1}y = x^{n+1}y$ and $(x - x^{n+1}y)^n = 0$. Therefore x is the sum of $x^{n+1}y \in S$ and $x - x^{n+1}y \in N$.

2) $\Rightarrow$ 1). Let $x = u + a$, where $u \in S$, $a \in N$ and $ua = au$. Again by [1, Lemma 1], there exists a $t \in R$ such that $u^2t = u$, $ut = tu$ and $at = ta$. Since $x - x^2t = (u + a) - (u + a)^2t = a - a(2u + a)t \in N$ and $xt = tx$, we conclude that x is strongly π -regular.

(2) As was claimed in [4, Remark], every homomorphic image of R is normal and π -regular. Since every primitive local ring is a division ring, Lemma 1 shows that R/J is a subdirect sum of division rings and reduced.

Hence J coincides with N and R/J is strongly regular by (1). As is well known, R is a subdirect sum of subdirectly irreducible rings, and so the latter assertion is clear by Lemma 1.

Proof of Theorem 3. (1) By a theorem of Chacron (see, e.g., [2, Theorem 1]), 2) implies 1).

1) $\Rightarrow$ 3). Let x be an arbitrary element of R . Then there exists a positive integer k such that $x^{2^k} = x^k$. It is easy to see that $x = x^{k+1} + (x - x^{k+1})$, $(x^{k+1})^{k+1} = x^{k+1}$ and $(x - x^{k+1})^k = 0$.

3) $\Rightarrow$ 2). Let $x = u + a$, where $u \in E_q$ for some $q > 1$, $a \in N$ and $ua = au$. Since $x - x^q = (u + a) - (u + a)^q = a + ay$ with some $y \in \langle u, a \rangle$, we see that $x - x^q \in N$, proving 2).

(2) This is an easy consequence of Theorem 2 (2) and Jacobson's commutativity theorem.

Corollary 1. *Let R be a normal ring. If every $x \in R$ can be written in the form $x = u + a$, where $u \in E_q$, $a \in N$ and $ua = au$, then N forms an ideal, R/N is a subdirect sum of finite fields of order at most q , and R is a subdirect sum of nil rings and/or local rings R_i such that $R_i/N(R_i)$ is a finite field of order at most q .*

As is well known, if every $x \in R$ may be written in at most one way in the form $x = e + a$, where $e \in E$ and $a \in N$, then R is normal. Hence we have

Corollary 2 (A. Yaquib). *Suppose every element x of R can be written uniquely in the form $x = e + a$, where $e \in E$ and $a \in N$. Then N forms an ideal, R/N is Boolean, and R is a subdirect sum of nil rings and/or local rings R_i such that $R_i/N(R_i)$ is $\text{GF}(2)$.*

Remark 1. Suppose every element x in a periodic ring R may be written in at most one way in the form $x = u + a$, where $u \in E_\infty$ and $a \in N$, then $R = E_\infty \oplus N$ by [3, Theorem 3]. Now, let $R = \mathbf{Z}/4\mathbf{Z}$. Then R satisfies the hypothesis in Corollary 2, but E does not form an ideal.

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