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ON PROJECTIVE DIFFEOMORPHISMS NOT NECESSARILY PRESERVING COMPLEX STRUCTURE

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Introduction. In his recent paper [5], Y. Tashiro has investigated conformal diffeomorphisms which do not necessarily preserve product structure between locally product Riemannian manifolds and determined the structure tensnrs on the manifolds. In connection with this problem, in 1959, he had early solved the corresponding problem on projective diffeomorphisms in [3, 4]. On the other hand, in 1941, N. Coburn [1] proved that a projective diffeomorphism f of a Kaehlerian manifold M onto a Kaehlerian manifold M^* which preserves the complex structure is affine. However projective diffeomorphisms between Kaehlerian manifolds which do not necessarily preserve complex structure have not been investigated yet.

The purpose of the present paper is to show generalizations of Coburn's theorem. § 1 will be devoted to give some formulae and lemmas used later. In § 2 we shall consider the problem in two different directions and give some corollaries. Last of all, in § 3, motivated from a theorem due to S. Tachibana [2] on the infinitesimal projective trans formation, we shall consider projective diffeomorphisms under an assumption analogous to the theorem.

The summation convention is used throughout this paper and indices run on the following ranges;

$$h,\ i,\ j,\ k,\ \cdots = 1,\ 2,\ 3,\ \cdots,\ m,$$

m being the topological dimension of M.

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1. Almost complex manifolds and formulae. Let (M, g) and (M^*, g^*) be Riemannian manifolds with Riemannian structure g and g^* respectively and suppose that there is given a projective diffeomorphism f of M onto M^*. We shall denote the induced tensors on M by f by the same letters as the original tensors on M^*. Then by the definition of projective diffeomorphism the Christoffel symbols $\left\{ {h \atop ji} \right\}$ and $\left\{ {h \atop ji} \right\}^*$ formed by g and g^* respectively are related as follows:

$$\left\{ {h \atop ji} \right\}^* = \left\{ {h \atop ji} \right\} + p_j\delta_i^h + p_i\delta_j^h \tag{1.1}$$

where p_j is a gradient vector field on M. The diffeomorphism f is affine if the vector field p_j identically vanishes.

Let D be Riemannian connection, and $K_{kji}{}^h$ the curvature tensor of M, and indicate quatities of M^* by asterisking. It is well known that the following equations are valid on M:

$$K^*{}_{kji}{}^h = K_{kji}{}^h - \delta_j^h p_{ki} - \delta_k^h p_{ji} \tag{1.2}$$

$$D_j g^*{}_{ih} = 2p_j g^*{}_{ih} + p_i g^*{}_{jh} + p_h g^*{}_{ij} \tag{1.3}$$

where p_{ji} is given by

$$p_{ji} = D_j p_i - p_j p_i\,. \tag{1.4}$$

Next, let (M^*, g^*, G) be an almost Hermitian manifold with almost complex structure G. Then, we have equations

$$G_j{}^t G_t{}^h = -\delta_j^h \tag{1.5}$$

$$G_j{}^t G_i{}^s g^*{}_{ts} = g^*{}_{ji} \tag{1.6}$$

$$G^*{}_{ji} = -G^*{}_{ij} \tag{1.7}$$

where we have put $G^*{}_{ji} = G_j{}^t g^*{}_{ti}$, see [6]. The manifold M^* is a K-space if the fundamental 2-form is a Killing tensor, that is, the equation

$$D^*{}_j G_i{}^h + D^*{}_i G_j{}^h = 0 \tag{1.8}$$

holds, or M^* is a Kaehlerian manifold if the equation

$$D^*{}_j G_i{}^h = 0 \tag{1.9}$$

or equivalently

$$D^*{}_j G^*{}_{ih} = 0 \tag{1.10}$$

holds. A Kaehlerian manifold is a K-space, and a K-space is a Kaehlerian manifold if the almost complex structure is integrable. We have the following

Lemma 1. *If there exists a projective diffeomorphism f of a Riemannian manifold (M, g) onto a Kaehlerian manifold (M^*, g^*, G), we have the equations*

$$D_j G_i{}^h = p_i G_j{}^h - p_t G_i{}^t \delta_j^h \tag{1.11}$$

$$D_j G^*{}_{ih} = 2p_j G^*{}_{ih} + p_i G^*{}_{jh} + p_h G^*{}_{ij} \tag{1.12}$$

$$K_{kji}{}^{h}G_{i}{}^{t} - K_{kjt}{}^{t}G_{t}{}^{h} = G_{j}{}^{h}p_{ki} - G_{k}{}^{h}p_{ji} - G_{i}{}^{t}(\delta_{j}^{h}p_{kt} - \delta_{k}^{h}p_{jt}) \tag{1.13}$$

$$D_{j}G_{i}{}^{h} + D_{i}G_{j}{}^{h} = p_{i}G_{j}{}^{h} + p_{j}G_{i}{}^{h} - p_{t}G_{i}{}^{t}\delta_{j}^{h} - p_{t}G_{i}{}^{t}\delta_{i}^{h} \tag{1.14}$$

and

$$D_{t}G_{j}{}^{t} = -mp_{t}G_{j}{}^{t}. \tag{1.15}$$

If there exists a projective diffeomorphism f *of a Riemannian manifold* (M, g) *onto a K-space* (M^{*}, g^{*}, G), *we have the equations* (1. 14) *and* (1. 15).

Proof. If (M^{*}, g^{*}, G) is an almost Hermitian manifold, it follows from (1. 1) that

$$D^{*}{}_{j}G_{i}{}^{h} = D_{j}G_{i}{}^{h} - (p_{i}G_{j}{}^{h} - p_{t}G_{i}{}^{t}\delta_{j}^{h})$$
$$D^{*}{}_{j}G^{*}{}_{ih} = D_{j}G^{*}{}_{ih} - (2p_{j}G^{*}{}_{ih} + p_{i}G^{*}{}_{jh} + p_{h}G^{*}{}_{ij}).$$

By substitution of these equations into (1. 8), (1. 9) or (1. 10), (1. 11), (1. 12), (1. 14) and (1. 15) are obtained. Applying Ricci's formula to (1. 11) and by straightforward computation, we have easily (1. 13). Q. E. D.

Now, let (M, g, F) and (M^{*}, g^{*}, G) be almost complex manifolds with almost complex structures F and G respectively. If there exists a diffeomorphism f of M on to M^{*}, G defines an almost complex structure $f^{*}(G)$ on M induced by f^{*}. We can define a new endomorphism $H=f^{*}(G)F$ on the tangent space of M by the composition of endomorphisms $f^{*}(G)$ and F. We define a scalar field τ on M by

$$\tau = \mathrm{Tr}(H), \tag{1.16}$$

where Tr means the trace of the endomorphism.

Lemma 2. *Let* (M, g, F) *and* (M^{*}, g^{*}, G) *be almost complex manifolds of real dimension* m *and suppose that there exists a diffeomorphism* f *of* M *onto* M^{*}.

(i) *If* $f^{*}(G) = \pm F$ *we have* $\tau = \mp m$ *respectively.*

(ii) *If* $f^{*}(G)$ *is commutative with* F *and* $f^{*}(G) \neq \pm F$, *then* τ *is constant and* $\tau \neq \pm m$. *Moreover,* H *defines an almost product structure on* M.

(iii) *If* $f^{*}(G)$ *is anti-commutative with* F, *we have* $\tau = 0$.

Proof. In this proof, we write G instead of $f^{*}(G)$ for simplicity.

In the case (i), we have $GF = \pm F^{2} = \mp I$. Hence $\tau = \mathrm{Tr}(GF) = \mp\mathrm{Tr}(I) = \mp m$.

To prove (ii), in the first place, let V be the tangent space of M at an arbitrary fixed point $x \in M$, and denote the linear transformation on V induced by F and G by the same letters. Let V^c be the complexified vector space of V. Since $G \neq \pm F$, V^c splits to the direct sum

$$V^c = V^{++} \oplus V^{+-} \oplus V^{-+} \oplus V^{--} \tag{1.17}$$

where, for instance, the space V^{+-} is the intersection of the eigenspace of G belonging to the eigenvalue $+\sqrt{-1}$ and the eigenspace of F belonging to the eigenvalue $-\sqrt{-1}$. It is clear that $\dim_C(V^{++}) = \dim_C(V^{--}) = a$, $\dim_C(V^{+-}) = \dim_C(V^{-+}) = b$ and $2(a+b) = m$, where a and b are integers. G and F can be extended to complex-linear transformations on V^c. Denote them by $\tilde{G}$ and $\tilde{F}$ and put $\tilde{H} = \tilde{G}\tilde{F}$. Since $GF = FG$, it holds $\tilde{G}\tilde{F} = \tilde{F}\tilde{G}$. Choosing a basis of V^c composed of bases of the subspaces in the direct decomposition (1.17), we then see that, with respect to this basis, $\tilde{H}$ has the form

$$\tilde{H} = \begin{pmatrix} -I(a) & & & 0 \\ & +I(b) & & \\ & & +I(b) & \\ 0 & & & -I(a) \end{pmatrix}$$

$I(n)$ being the identity matrix of degree n. Thus we have $\mathrm{Tr}(\tilde{H}) = 2(b-a)$. If we choose the above basis as a real vector space for V, $H = GF$ is given by the same form as $\tilde{H}$. Therefore we have $\tau = \mathrm{Tr}(H) = 2(b-a)$ at $x \in M$. Since a and b are integers and τ is continuous on M, τ is a constant and $\tau = 2(b-a)$ on M. It is easy to see that $a \neq 0$, m unless $G = \pm F$. This proves $\tau \neq \pm m$. In this case, $H^2 = I$ and $H \neq \pm I$, that is, H is an almost product structure on M.

In the case (iii), $GF = -FG$ and thus

$$\tau = \mathrm{Tr}(GF) = \mathrm{Tr}(-FG) = -\mathrm{Tr}(FG) = -\tau.$$

We have $\tau = 0$. Q. E. D.

2. Generalizations of Coburn's theorem.

Theorem 3. (a) *Let (M, g, F) be a K-space, and (M^*, g^*, G) a Kaehlerian manifold. Then a projective diffeomorphism f of M onto M^* is affine if $f^*(G)$ is commutative with F.*

(b) *Let (M, g, F) and (M^*, g^*, G) be K-spaces. Then a projective diffeomorphism f of M onto M^* is affine if $f^*(G)$ is anti-commutative with F.*

Proof. In the case (a), we have

$$F_j{}^t G_t{}^h = G_j{}^t F_t{}^h \tag{2.1}$$

and τ is constant from Lemma 2 (i) and (ii). Moreover, since M is a K-space the equations

$$D_j F_i{}^h = -D_i F_j{}^h \tag{2.2}$$

$$D_t F_i{}^t = 0 \tag{2.3}$$

hold. Substituting (1.11) into $D_j\tau = 0$ and taking account of (2.1), we obtain

$$(D_j F_s{}^t) G_t{}^s = 0. \tag{2.4}$$

If we transvect (2.4) with $G_i{}^j$ and use (2.2), (2.1), (1.15), (2.3), (2.1), (1.5), (1.11) and $F_t{}^t = 0$ in this order, we see

$$\begin{aligned} 0 &= G_i{}^j (D_j F_s{}^t) G_t{}^s \\ &= -G_i{}^j (D_s F_j{}^t) G_t{}^s \\ &= -G_i{}^j D_s (F_j{}^t G_t{}^s) + G_i{}^j F_j{}^t D_s G_t{}^s \\ &= -G_i{}^j D_s G_j{}^t F_t{}^s - m p_s G_i{}^j G_j{}^t F_t{}^s \\ &= -G_i{}^j (p_j G_s{}^t - p_r G_j{}^r \delta_s^t) F_t{}^s + m p_s F_i{}^s \\ &= -\tau p_j G_i{}^j + m p_s F_i{}^s \end{aligned}$$

and thus we have

$$\tau p_s G_i{}^s = m p_s F_i{}^s . \tag{2.5}$$

If $f^*(G) \neq \pm F$, then it follows from Lemma 2 (ii) that $\tau \neq \pm m$. On the other hand from (2.5) and (2.1) we have the equation

$$(m^2 - \tau^2) p_j = 0 .$$

Hence we have $p_j = 0$ and f is affine. If $f^*(G) = \pm F$, then we observe $\tau = \mp m$ and $f^*(G)F = \mp I$. By account of these properties it follows from (2.5) that f is affine.

In the cace (b), we have

$$G_j{}^t F_t{}^h = -F_j{}^t G_t{}^h \tag{2.6}$$

and $\tau = 0$. If we apply the operator D_h to (2.6) and take account of (2.2) and (2.3), we have

$$(D_h G_j{}^t) F_t{}^h = -(D_h F_j{}^t) G_t{}^h - F_j{}^t D_h G_t{}^h$$

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$$= (D_j F_h{}^t) G_t{}^h - F_j{}^t D_h G_t{}^h$$
$$= -(D_j G_h{}^t) F_t{}^h - F_j{}^t D_h G_t{}^h .$$

Substituting (1. 15) into the last equation, we obtain

$$(D_s G_j{}^t + D_j G_s{}^t) F_t{}^s = m p_s G_t{}^s F_j{}^t . \tag{2. 7}$$

On the other hand, since M^* is a K-space, it follows from (1. 14) transvected with $F_j{}^h$ that

$$(D_s G_j{}^t + D_j G_s{}^t) F_t{}^s = -2 p_s G_t{}^s F_j{}^t .$$

Putting this expression equal to (2. 7), we get

$$(m + 2) p_s G_t{}^s F_j{}^t = 0 .$$

Thus, $p_j = 0$ and consequently f is affine. Q. E. D.

Corollary 4. *In addition to the assumption of Theorem* 3 (a), *suppose that* $f^*(G) \neq \pm F$. *Then a necessary and sufficient condition for* M *to be a locally product manifold with the structure tensor* $H = f^*(G) F$, *is that the K-space structure* (g, F) *on* M *is Kaehlerian.*

Proof. By Theorem 3 (a), we have $D_j G_i{}^h = 0$. Since (M, g, F) is a K-space and $GF = FG$, we obtain

$$H_j{}^t D_t H_i{}^h = -F_j{}^t D_t F_i{}^h . \tag{2. 8}$$

By Lemma 2 (ii), H is an almost product structure on M. If we denote by $N(H)$ and $N(F)$ the Nijenhuis tensors of the tensors H and F respectively, then the relation

$$N(H) = - N(F) \tag{2. 9}$$

follows immediately from (2. 8). Therefore $N(H) = 0$ is equivalent to $N(F) = 0$, that is, the integrability of the almost product structure H is equivalent to that of the almost complex structure F. Q. E. D.

Since a Kaehlerian manifold is a K-space, we have the following

Corollary 5. *Let* (M, g, F) *and* (M^*, g^*, G) *be Kaehlerian manifolds. Then a projective diffeomorphism* f *of* M *onto* M^* *is affine if one of the following conditions is satisfied* :

(a) $f^*(G)$ *is commutative with* F.

(b) $f^*(G)$ *is anti-commutative with* F.

Moreover, if (a) *is satisfied and* $f^*(G) \neq \pm F$, *then both* M *and* M^* *are locally product manifolds.*

In Corollary 5 (a), especially if f satisfies the condition $f^*(G) = \pm F$, this result turns Coburn's theorem. On the other hand, if f preserves the complex structure, the Riemannian structure $f^*(g^*)$ on M induced by g^* on M^* becomes a Hermitian structure endowed with F. We can then state another generalization of Coburn's theorem as follows:

Theorem 6. *Let* (M, g, F) *be a Kaehlerian manifold, and* (M^*, g^*) *another Riemannian manifold. If a projective diffeomorphism* f *of* M *onto* M^* *makes the induced Riemannian structure* $f^*(g^*)$ *to be a Hermitian structure endowed with* F, *then* f *is affine.*

Proof. By the assumption

$$g^*_{ji} = F_j{}^t F_i{}^s g^*_{ts}. \tag{2.10}$$

If we apply the operator D_k to (2. 2) and take account of $D_j F_i{}^h = 0$, we have

$$D_k g^*_{ji} = F_j{}^t F_i{}^s D_k g^*_{ts}. \tag{2.11}$$

Substituting (1. 3) into (2. 11) and using (2. 10), we see

$$p_j g^*_{ki} + p_i g^*_{jk} = p_t F_j{}^t F_i{}^s g^*_{ks} + p_s F_i{}^s F_j{}^t g^*_{tk}.$$

If we transvect this with g^{*ki} and use the identity $F_t{}^t = 0$, we can easily get $p_j = 0$; this proves the theorem. Q. E. D.

We shall conclude this section with a consequence which is obtained in the same way as th proof of Theorem 6. Let (M, g, F) and (M^*, g^*, G) be Kaehlerian manifolds. Then $\hat{G}^* = (1/2)G_{ji}{}^* dx^j \wedge dx^i$ is the funadmental 2-form on M^* and the form $C_F(\hat{w})$ for a 2-form $\hat{w} = w_{ji} dx^j \wedge dx^i$ is defined by $C_F(\hat{w}) = w_{ji} F_t{}^j F_s{}^i dx^t \wedge dx^s$ in terms of real coordinate system. Using (1. 12) instead of (1. 3), we have the following

Theorem 7. *Let* (M, g, F) *and* (M^*, g^*, G) *be Kaehlerian manifolds. Then a projective diffeomorphism* f *of* M *onto* M^* *is affine if one of the following conditions is satisfied:*

(a) $C_F(f^*(\hat{G}^*)) = f^*(\hat{G}^*)$ *and* $m > 2$.

(b) $C_F(f^*(\hat{G}^*)) = -f^*(\hat{G}^*)$.

3. A theorem under the condition $\hat{K}^* = 0$. In a Kaehlerian manifold (M^*, g^*, G), the Chern 2-form $\hat{K}^*$ is a closed form defined by

$$\hat{K}^* = \hat{K}^*{}_{ji} dx^j \wedge dx^i \tag{3.1}$$

$$\hat{K}_{ji} = 2K^*{}_{jits}G^{*ts} = -2K^*{}_{jit}{}^s G_s{}^t \quad (= -\hat{K}^*{}_{ij}) \tag{3.2}$$

where we have put $G^{*kh} = G_j{}^h g^{*kj}$.

Theorem 8. *Let (M, g) be a compact Riemannian manifold of non-negative scalar curvature $k \geq 0$, and (M^*, g^*, G) a Kaehlerian manifold with vanishing Chern 2-form $\hat{K}^* = 0$. Then a projective diffeomorphism f of M onto M^* is affine. Moreover, M has necessarily the vanishing scalar curvature $k = 0$.*

Proof. By the assumption, (1.13) holds. Putting $G_{ji} = G_j{}^h g_{hi}$, we have

$$\begin{aligned} & K_{kjsh} G_i{}^s - K_{kji}{}^s G_{sh} \\ & = p_{ki} G_{jh} - p_{ks} G_i{}^s g_{jh} - p_{ji} G_{kh} + p_{sj} G_i{}^s g_{kh}. \end{aligned} \tag{3.3}$$

Transvecting this with g^{ki} we have

$$K^t{}_{jsh} G_t{}^s + K_j{}^s G_{sh} = (g^{st} p_{st}) G_{jh} - (p_{st} G^{ts}) g_{jh} - p_j{}^s G_{sh} + p_{js} G_h{}^s. \tag{3.4}$$

Again transvecting this with g^{jh} and taking account of $p_{ji} = p_{ij}$ and $G_{jh} g^{jh} = G_t{}^t = 0$, we obtain

$$p_{st} G^{st} = 0. \tag{3.5}$$

If we substitute (3.5) into (3.4), we have

$$K^t{}_{jsh} G_t{}^s + K_j{}^s G_{sh} = (g^{st} p_{st}) G_{jh} - p_j{}^s G_{sh} + p_{js} G_h{}^s. \tag{3.6}$$

On the other hand, from (1.2) and $G_t{}^t = 0$, we obtain

$$\begin{aligned} K^t{}_{jsh} G_t{}^s &= K_{hsj}{}^t G_t{}^s \\ &= \{K^*{}_{hsj}{}^t - (\delta_s^t p_{hj} - \delta_h^t p_{sj})\} G_t{}^s \\ &= K^*{}_{hsj}{}^t G_t{}^s + p_{sj} G_h{}^s. \end{aligned}$$

Substituting this into (3.6), we get

$$K^*{}_{hsj}{}^t G_t{}^s + K_j{}^s G_{sh} = (g^{st} p_{st}) G_{jh} - p_j{}^s G_{sh}. \tag{3.7}$$

If we transvect this with G^{hj} and take account of $G^{hj} G_{sh} = G_h{}^j G_s{}^h = -\delta_s^j$, we have

$$K^*{}_{hjs}{}^t G_t{}^s G^{hj} - k = -(m-1) g^{st} p_{st}. \tag{3.8}$$

On the other hand, it follows from the first Bianchi's identity that

$$K^*{}_{hsj}{}^tG_t{}^s = G^{*ts}K^*{}_{hsjt} = -(1/4)\hat{K}^*{}_{hj}. \tag{3.9}$$

Since $\hat{K}^* = 0$ on M^*, (3.9) becomes $K^*{}_{hsj}{}^tG_t{}^s = 0$. If we substitute this into (3.8) we have consequently

$$k - (m-1)g^{st}p_{st} = 0, \tag{3.10}$$

or, by making use of the definition (1.4) of p_{ji},

$$k - (m-1)(\Delta p - g^{ji}p_jp_i) = 0$$

where Δ is the Laplacian operator with respect to g. Since M is compact, applying Green's theorem, we have

$$\int_M [k + (m-1)g^{ji}p_jp_i] *1 = 0 \tag{3.11}$$

where $*1$ is the volume element on M. By the assumption, $k \geq 0$ and $g^{ji}p_jp_i \geq 0$, (3.11) implies $k = 0$ and $p_j = 0$. This is the desired result. Q. E. D.

REFERENCES

[1] N. COBURN: Unitary spaces with corresponding geodesics, Bull. Amer. Math. Soc. **47** (1941), 901—910.

[2] S. TACHIBANA: On infinitesimal conformal and projective transformations of compact K-space, Tôhoku Math. J. **13** (1961), 386—392.

[3] Y. TASHIRO: On projective transformations of Riemannian manifolds, J. Math. Soc. Japan **11** (1959), 196—204.

[4] Y. TASHIRO: Conformal Transformations in Complete Riemannian Manifolds, Publ. S. G. G. **3**, 1967.

[5] Y. TASHIRO: On product Riemannian manifolds and conformal diffeomorphisms, to appear.

[6] K. YANO: Differential Geometry on Complex and Almost Complex Spaces, Pergamon Press, 1965.

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