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Some commutativity properties for rings

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SOME COMMUTATIVITY PROPERTIES FOR RINGS

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Let A be a non-empty subset of the ring R with center C; let N denote the set of nilpotent elements of R, and $V_R(A)$ the centralizer of A in R. Let q be a fixed integer greater than 1. We consider the following seven properties:

(I-A) For each $x \in R$, there exists a polynomial $f(\lambda)$ in $\boldsymbol{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(I′-A) For each $x \in R$, either $x \in C$ or there exists a polynomial $f(\lambda)$ in $\boldsymbol{Z}[\lambda]$ such that $x - x^2 f(x) \in A$.

(II-A)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or x and y both belong to $V_R(A)$.

(II′-A)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or $[x,y] = 0$.

(II″-A)$_q$ If $x, y \in R$ and $x - y \in A$, then either $x^q = y^q$ or $[xy, yx] = 0$.

(III-A) For every $a \in A$ and $x \in R$, $[[a,x],x] = 0$.

(IV-A) If $a \in A$, $x \in R$ and $[a,x]^2 = 0$, then $[a,x] \in C$.

Needless to say, (I-A) implies (I′-A), (II-A)$_q$ does (II′-A)$_q$, and (II′-A)$_q$ does (II″-A)$_q$. The major purpose of this paper is to prove the following

Theorem 1. *The following statements are equivalent:*

1) *R is commutative.*

2) *There exists a (multiplicatively) commutative subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (III-A).

3) *There exists a commutative subset A for which R satisfies* (I-A), (II-A)$_q$ *and* (IV-A).

4) *There exists a commutative subset A of N for which R satisfies* (I′-A) *and* (II″-A)$_2$.

5) *There exists a commutative subset A of N for which R satisfies* (I′-A) *and* (III-A).

6) *There exists a commutative subset A of N such that R satisfies* (I′-A) *and* (IV-A).

In preparation for proving Theorem 1, we establish the following two lemmas.

Lemma 1. (1) *Let ψ be a ring homomorphism of R onto R^*. If R satisfies* (I-A), (I′-A), (II-A)$_q$, (II′-A)$_q$, (II″-A)$_q$ *or* (III-A), *then R^* satisfies* (I-$\psi(A)$), (I′-$\psi(A)$), (II-$\psi(A)$)$_q$, (II′-$\psi(A)$)$_q$, (II″-$\psi(A)$)$_q$ *or* (III-$\psi(A)$), *respectively.*

(2) *If R satisfies* (I′-A), *then N is contained in $A^+ + C$, where A^+ is the additive subsemigroup generated by A.*

(3) *If R satisfies* (II-A)$_q$, *then $[a,x^q] = 0$ for all $a \in A$ and $x \in R$.*

(4) *If R satisfies* (I′-A) *and* (II″-A)$_q$ (*resp.* (I′-A) *and* (III-A)), *then R is normal, that is, every idempotent e of R is central.*

(5) *If A is commutative and R satisfies* (I′-A), *then N is a commutative nil ideal of R containing the commutator ideal of R and is contained in $V_R(A)$.*

(6) *If A is commutative and R satisfies* (I′-A), *then* (IV-A) *implies* (III-A), *and* (III-A) *does* (II″-A)$_q$.

Proof. (1) Straightforward.

(2) By a trivial induction on nilpotency index.

(3) Suppose that $[a,x] \neq 0$ for some $a \in A$ and $x \in R$. Since $(x+a)^q = x^q$ by (II-A)$_q$, we have $[a,x^q] = [x+a,x^q] = [x+a,(x+a)^q] = 0$.

(4) Given $x \in R$, we set $a = xe - exe$. Since $a^2 = 0$, (I′-A) shows that $a \in C \cup A$. Then, by (II″-A)$_q$ (resp. (III-A)), $e+a = (e+a)^q = e^q = e$ or $a = [(e+a)e, e(e+a)] = 0$ (resp. $a = [[a,e],e] = 0$). Thus in any case $a = 0$, and hence $xe = exe$. Similarly, we can show that $ex = exe$, and therefore $xe = ex$.

(5) By (2) and a theorem of Chacron (see, e.g. [5, Theorem 1]).

(6) By (5), N is a commutative ideal containing the commutator ideal of R and is contained in $V_R(A)$. Hence, in case (IV-A) is satisfied, for any $a \in A$ and $x \in R$ we have $[a,x]^2 = [a,x[a,x]] - x[a,[a,x]] = 0$. Then, $[a,x]$ is central by (IV-A), and therefore R satisfies (III-A). On the other hand, if (III-A) is satisfied then $[x(x-a),(x-a)x] = [[a,x],x^2] = 0$ by (5) and (III-A). This means that if $x-y \in A$ then $[xy,yx] = 0$, and R satisfies (II″-A)$_q$.

Lemma 2. *Let R be a normal, subdirectly irreducible ring. If A is a commutative subset of N not contained in C for which R satisfies* (I′-A), *then R is of characteristic p^α, where p is a prime and $\alpha > 0$.*

Proof. Choose $a \in A$ and $b \in R$ with $[a,b] \neq 0$. By (I′-A), $b - b^2 f(b) \in A$ for some $f(\lambda) \in \boldsymbol{Z}[\lambda]$, and hence $b^m = b^{2m} b_0^m$ with some $m > 0$ and some b_0 in the subring $\langle b \rangle$ generated by b. Since $b \notin N$ by Lemma 1 (5),

$e=(bb_0)^m$ is a non-zero central idempotent, and therefore e is the identity element 1 of the subdirectly irreducible ring R and b is invertible. Moreover, b^{-1} is integral over $\boldsymbol{Z}\cdot 1$. Since a cannot commute with both $2b^{-1}$ and $3b^{-1}$, there exists an integer $k>1$ such that $[a,kb^{-1}]\neq 0$. Then, by the above argument, $(kb^{-1})^{-1}$ is integral over $\boldsymbol{Z}\cdot 1$, and hence $k^{-1}\cdot 1=(kb^{-1})^{-1}b^{-1}$ also is integral over $\boldsymbol{Z}\cdot 1$. Obviously, this implies that the additive order of 1 is non-zero, and therefore the subdirectly irreducible ring R is of characteristic p^{α}, where p is a prime and $\alpha>0$.

We are now in a position to complete the proof of Theorem 1.

Proof of Theorem 1. Obviously, 1) $\Rightarrow$ 3) and 6). By Lemma 1 (6), 3) $\Rightarrow$ 2) and 6) $\Rightarrow$ 5) $\Rightarrow$ 4).

2) $\Rightarrow$ 1). In view of Lemma 1 (1), we may (and shall) assume that R is subdirectly irreducible. According to [3, Theorem 19] and (I-A), it suffices to show that $A\subseteq C$. Suppose, to the contrary, that there exist $a\in A$ and $b\in R$ such that $[a,b]\neq 0$. By (I-A) and (II-A)$_q$, $b^q=(b^2f(b))^q$ with some $f(\lambda)\in \boldsymbol{Z}[\lambda]$. Since $b\notin N$ by Lemma 1 (2), Lemma 1 (4) shows that $e=(bf(b))^q$ is a non-zero central idempotent, and hence e is the identity element 1 of the subdirectly irreducible ring R. By (I-A), $2-2^2g(2)\in A$ with some $g(\lambda)\in\boldsymbol{Z}[\lambda]$. Thus, we can find a non-zero integer k such that $k=k\cdot 1\in A$. Obviously, $[a,b+ik]\neq 0$ for all $i\in\boldsymbol{Z}$. Hence, by (II-A)$_q$, every $b+ik$ is a zero of the polynomial $(\lambda+k)^q-\lambda^q$. Note here that R/N is a subdirect sum of commutative integral domains (Lemma 1 (5)). Then, since $b+ik$ ($i=0,1,\cdots,q$) are zeros of $(\lambda+k)^q-\lambda^q$, we can easily see that $q!k^q\in N$, and so $h\cdot 1=0$ for some positive integer h. This means that the characteristic of the subdirectly irreducible ring R is p^{α}, where p is a prime and $\alpha>0$. We set $q=p^{\beta}t$, where $(p,t)=1$. Noting here that every non-zero idempotent of $\bar{R}=R/N$ coincides with $\bar{1}$ (Lemma 1 (4)), we can easily see that $\langle\bar{b}\rangle=\mathrm{GF}(p^{\gamma})$ with some $\gamma>0$, and therefore $b^{q^{\gamma}}-b^{t^{\gamma}}\in N\subseteq V_R(A)$ (Lemma 1 (5)). Combining this with $[a,b^{q^{\gamma}}]=0$ (Lemma 1 (3)), we have $[a,b^{t^{\gamma}}]=[a,b^{q^{\gamma}}]-[a,b^{q^{\gamma}}-b^{t^{\gamma}}]=0$. Now, by (III-$A$), $t^{\gamma}b^{t^{\gamma}-1}[a,b]=[a,b^{t^{\gamma}}]=0$. Then the usual argument of replacing b by $b+1$, etc. shows that $t^{\gamma}[a,b]=0$. Since $p^{\alpha}[a,b]=0$ and $(p,t)=1$, it forces a contradiction $[a,b]=0$.

4) $\Rightarrow$ 1). First, we claim that $[x^2,[x,a]]=0$ for all $a\in A$ and $x\in R$. In fact, by (II$''$-A)$_2$ and Lemma 1 (5), either

$$[x^2,[x,a]]=[(x+a)x,x(x+a)]=0,$$

or $(x+a)^2=x^2$ and hence

$$[x^2,[x,a]] = [x,[x^2,a]] = [x,[x^2,x+a]] = [x,[(x+a)^2,x+a]] = 0.$$

In view of Lemma 1 (1), we may (and shall) assume that R is subdirectly irreducible. According to [3, Theorem 19] and (I′-A), it suffices to show that $A \subseteq C$. Suppose, to the contrary, that there exist $a \in A$ and $b \in R$ such that $[a,b] \neq 0$. Then, by Lemma 1 (4) and Lemma 2 (and its proof), R is of characteristic p^α (p a prime and $\alpha > 0$), and $\bar{b} = b+N$ is algebraic over GF(p) (Lemma 1 (5)). Furthermore, noting that every non-zero idempotent of R/N coincides with $\bar{1}$ (Lemma 1 (4)), we can easily see that $\langle \bar{b} \rangle = \mathrm{GF}(p^\beta)$ with some $\beta > 0$, and therefore $b^{p^\gamma} - b \in N$ for some $\gamma \geq \alpha$. Now, by the opening claim,

$$2[b,[b,a]] = [(b+1)^2,[b+1,a]] - [b^2,[b,a]] = 0.$$

We claim further that $[b,[b,a]] = 0$. Since R is of characteristic p^α, it suffices to consider the case $p = 2$. Then $[b^{2^\gamma} - b,[b,a]] = 0$ by Lemma 1 (5). On the other hand, $[b^2,[b,a]] = 0$ implies $[b^{2^\gamma},[b,a]] = 0$, and therefore $[b,[b,a]] = 0$ holds always. Combining the last claim with $[b^{p^\gamma} - b,a] = 0$ (Lemma 1 (5)), we obtain

$$[b,a] = [b^{p^\gamma},a] - [b^{p^\gamma} - b,a] = p^\gamma b^{p^\gamma - 1}[b,a] = 0$$

This contradiction proves that R is commutative.

Corollary 1. *The following statements are equivalent* :

1) *R is commutative.*

2) *There exists a (multiplicatively) commutative additive subsemigroup A for which R satisfies* (I-A), (II′-A)$_q$ *and* (III-A).

3) *There exists a commutative additive subsemigroup A for which R satisfies* (I-A), (II′-A)$_q$ *and* (IV-A).

Proof. It suffices to show that, in case R satisfies (II′-A)$_q$ for a commutative additive subsemigroup A, $x-y \in A$ and $x^q \neq y^q$ imply $x \in V_R(A)$. Suppose, to the contrary, that there exists $a \in A$ such that $[a,x] \neq 0$. Then $(x+a)-x \in A$, $(x+a)-y \in A$ and $[x,y] = 0$. Since $[x+a,x] \neq 0$ and $[x+a,y] \neq 0$, we get $x^q = (x+a)^q = y^q$, a contradiction.

Corollary 2 (cf. [7, Theorem 1] and [4, Corollary 1]). (1) *If there exists a commutative subset A for which R satisfies* (I-A) *and* (II-A)$_2$, *then R is commutative.*

(2) *If there exists a commutative additive subsemigroup A for which R satisfies* (I-A) *and* (II′-A)$_2$, *then R is commutative.*

Proof. (1) In view of Lemma 1 (1), we may assume that R is subdirectly irreducible. Suppose that $[a,b] \neq 0$ for some $a \in A$ and $b \in R$. Then, as was shown in the proof of Theorem 1, R has 1. Since $(b+a)^2 = b^2$ and $(b+1+a)^2 = (b+1)^2$ by $(\mathrm{II}\text{-}A)_2$, we have

$$2a = \{b^2+2(b+a)+1\}-(b+1)^2 = (b+1+a)^2-(b+1)^2 = 0,$$

and therefore $[[a,b],b] = [a,b^2]+2b^2a-2bab = 0$ by Lemma 1 (3). Thus, we have seen that R satisfies (III-A), and R is commutative by Theorem 1.

(2) This is immediate by (1) and the proof of Corollary 1.

Corollary 3 (cf. [2, Theorem 2]). *Suppose that there exists a commutative subset A of N for which R satisfies* (I-A) *and* (III-A). *Then R is commutative.*

Remark 1. Theorem 1 is no longer valid if we remove the hypothesis (III-A) in 2) (resp. (IV-A) in 3)). A counterexample is given in [6, Remark, p. 18].

Remark 2. In [1, Theorem 1], the second author and Abu-Khuzam considered the following property:

(I*-N) for each $x \in R$, either $x \in C$ or there exists an integer $n > 1$ such that $x - x^n \in N$,

and proved that if N is commutative and R satisfies (I*-N) and (III-N), then R is commutative. Also, in [8, Theorem 1], the second author considered the following property:

(II″-A) if $x, y \in R$ and $x-y \in A$, then either $[xy,yx] = 0$ or x and y both belong to $V_R(A)$,

and proved that if there exists a commutative subset A of N for which R satisfies (I-A) and (II″-A), then R is commutative. We claim here that (II″-A) may be restated as follows: if $x-y \in A$ then $[xy,yx] = 0$. In fact, if $x-y \in A$ and $y \in V_R(A)$, then $[x,y] = [x-y,y] = 0$, and hence $[xy,yx] = 0$. Needless to say, Theorem 1 includes both [1, Theorem 1] and [8, Theorem 1].

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