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## On rings satisfying the identity $(x+x^2+\dots+x^n)^n=0$

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# ON RINGS SATISFYING THE IDENTITY

$$(x+x^2+\cdots+x^n)^{(n)}=0$$

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Throughout the present paper,  $R$  will represent a ring with center  $C$ . Let  $N$  be the set of nilpotents in  $R$ ,  $N^*$  the subset of  $N$  consisting of all  $a$  with  $a^2=0$ ,  $E$  the set of idempotents in  $R$ , and  $D$  the commutator ideal of  $R$ . For  $x \in R$ , we define inductively  $x^{(1)}=x$ ,  $x^{(k)}=x^{(k-1)} \circ x$ , where  $x \circ y = x+y+xy$ . We may write formally  $x^{(k)}=(1+x)^k-1$ .

Let  $n$  be a positive integer, and consider the following properties:

- (i)<sub>n</sub>  $(x+x^2+\cdots+x^n)^{(n)}=0$  for all  $x \in R$ .
  - (ii)  $N$  is commutative.
  - (ii)\*  $N^*$  is commutative.
  - (iii)  $[[a,x],x]=0$  for all  $a \in N$  and  $x \in R$ .
  - (iv) If  $a \in N$ ,  $x \in R$  and  $[a,x]^2=0$  then  $[a,x] \in C$ .
  - (\*) For any  $x, y \in R$ ,  $(x+xy) \circ (y+yx)=0$  if and only if  $x=y$ .
- If  $R$  has 1, then (i)<sub>n</sub> becomes
- (i)'<sub>n</sub>  $(1+x+x^2+\cdots+x^n)^n=1$  for all  $x \in R$ .

The present objective is to prove the following theorems.

**Theorem 1.** *Let  $R$  be a left  $s$ -unital ring satisfying (i)<sub>n</sub>. If  $R$  is normal (i.e.,  $E$  is central) then  $N$  is a nil ideal and  $R/N$  is commutative.*

**Theorem 2.** *Let  $R$  be a left  $s$ -unital ring satisfying (i)<sub>n</sub>.*

(1) *If  $R$  satisfies (ii)\*, then  $N$  is a nil ideal and  $R/N$  is a commutative regular ring.*

(2) *If  $R$  satisfies (ii), then  $N$  is a commutative nil ideal and  $R/N$  is a commutative regular ring.*

(3) *If  $R$  satisfies (ii) and (iii) (or (iv)), then  $R$  is commutative and  $R/N$  is a regular ring.*

**Theorem 3.** *Let  $R$  be a left  $s$ -unital ring satisfying (i)<sub>2\*</sub>. Then  $N$  is a nil ideal and  $R=R_1 \oplus R_2$ , where  $R_1$  is either 0 or a commutative regular ring of odd characteristic,  $R_2 \supseteq N$  and  $R_2/N$  is a Boolean ring.*

**Theorem 4.** *If  $R$  is a normal, left  $s$ -unital ring satisfying (i)<sub>2\*</sub> and*

(ii), then  $R$  is commutative and  $R = R_1 \oplus R_2$ , where  $R_1$  is either 0 or a regular ring of odd characteristic,  $R_2 \supseteq N$  and  $R_2/N$  is a Boolean ring.

**Theorem 5** (cf. [4, Theorem 2]). A left  $s$ -unital ring  $R$  satisfies  $(*)$  if and only if (a)  $R$  is commutative and  $R/N$  is a Boolean ring and (b)  $a^{(2)} = 0$  for all  $a \in N$ .

We start with the following lemma.

**Lemma 1.** Suppose that  $R$  satisfies  $(i)_n$  and  $p^a R = 0$ , where  $p$  is a prime. Then there exists a positive integer  $m$  such that  $x^m = x^{2^m}$  for all  $x \in R$ .

*Proof.* Let  $y = x + x^2 + \cdots + x^n$ , and  $n = p^\beta t$ , where  $\beta \geq 0$  and  $(p, t) = 1$ . Then there exists  $u(\lambda) \in \mathbb{Z}[\lambda]$  such that  $y^{(t)} = (1+y)^t - 1 = tx + x^2 u(x)$ . Noting here that  $(pR)^a = 0$ , we can easily see that  $(tx + x^2 u(x))^{p^\beta a} = 0$ . Because  $(t, p) = 1$ , we readily obtain  $x^{p^\beta a} - x^{p^\beta a+1} f(x) = 0$  with some  $f(\lambda) \in \mathbb{Z}[\lambda]$ . Now, by making use of the argument employed in the proof of [1, Lemma], we can find a positive integer  $m$  such that  $x^m = x^{2^m}$  for all  $x \in R$ .

**Remark 1.** (1) Recently, Komatsu [5] proved that a ring  $R$  with 1 satisfies a polynomial identity  $x^{2^m} - x^m = 0$  for some positive integer  $m$  if and only if the addition of  $R$  is equationally definable in terms of the multiplication and the successor operation.

(2) Let  $p$  be a prime. If  $(1+n)^n \equiv 1 \pmod{p}$  and  $p-1 \mid n$ , then  $\text{GF}(p)$  satisfies  $(i)_n$ . For instance,  $\text{GF}(3)$  satisfies  $(i)_4$ .

(3) Let  $R$  be a left  $s$ -unital ring satisfying  $(i)_n$ . Let  $x$  be an arbitrary non-zero element of  $R$ , and choose  $e \in R$  such that  $ex = x$ . Then, by  $(i)_n$ ,

$$0 = \{(1+e+e^2+\cdots+e^n)^n-1\}x = \{(n+1)^n-1\}x.$$

This means that the characteristic of  $R$  is non-zero. Furthermore,  $n$  has to be even. In fact, if  $n$  is odd, then

$$0 = \{(1-e+e^2-\cdots-e^n)^n-1\}x = -x,$$

which is a contradiction. In particular, if  $p$  is a prime and  $n = p^k$  then  $p = 2$ .

*Proof of Theorem 1.* By Remark 1(3), the characteristic of  $R$  is non-zero. Obviously, the hypothesis  $(i)_n$  and the normality are inherited by subrings, so we may assume that the additive group of  $R$  is a  $p$ -group, and

therefore  $p^a R = 0$  for some prime  $p$ . Then, by Lemma 1, there exists a positive integer  $m$  such that  $x^m = x^{2^m}$  for all  $x \in R$ , and so  $R$  satisfies the polynomial identity  $[x^m, y] = 0$ . Since  $[E_{11}, E_{11} + E_{12}] = E_{12} \neq 0$  in  $(GF(q))_2$  ( $q$  a prime),  $D$  is a nil ideal by [3, Proposition 2].

**Remark 2.** For  $e \in E$ , the following are equivalent :

- 1)  $e \in C$ .
- 2)  $[e, a] = 0$  for all  $a \in N^*$ .
- 3)  $[e, [e, a]] = 0$  for all  $a \in N^*$ .
- 4)  $[(e+a)e, e(e+a)] = 0$  for all  $a \in N^*$ .
- 5)  $[e, f] = 0$  for all  $f \in E$ .

In fact, it is clear that 1) implies 2)–5), and 2) does 3). In order to see that each of 3)–5) implies 1), given  $x \in R$ , we set  $a = ex(1-e) \in N^*$ . It is easy to see that if 3) or 4) is satisfied then  $a = 0$ , i.e.,  $ex = exe$ . Similarly, we can see that  $xe = exe$ . Finally, if 5) is satisfied, then  $e+a \in E$  and  $e+a = e(e+a) = (e+a)e = e$ , whence it follows again  $a = 0$ , i.e.,  $ex = exe$ ; similarly  $xe = exe$ . In particular,  $R$  is normal if and only if  $E$  is commutative, and (iii) implies the normality of  $R$ . Furthermore, if  $N^*$  is central then  $R$  is normal and  $N$  coincides with the prime radical of  $R$ . In fact, if  $a^n = 0$  and  $a^{n-k}(Ra)^{k+1} = 0$  then  $\{a^{n-k-1}(Ra)^{k+1}\}^2 \subseteq a^{n-k-1}Ra^{n-k}(Ra)^{k+1} = 0$ . Hence  $a^{n-k-1}(Ra)^{k+1} \subseteq C$ , and so  $a^{n-k-1}(Ra)^{k+2} = Ra^{n-k}(Ra)^{k+1} = 0$ . We get eventually  $(Ra)^{n+1} = 0$ .

In advance of proving Theorem 2, we state the next lemma.

**Lemma 2.** (1) If  $R$  satisfies (ii)\* then  $R/P$  is normal, where  $P$  is the prime radical of  $R$ .

(2) If a  $\pi$ -regular ring  $R$  satisfies (ii)\*, then  $N$  coincides with the Jacobson radical of  $R$  and  $R/N$  is strongly regular.

*Proof.* (1) Let  $\bar{e}$  be an arbitrary idempotent of  $\bar{R} = R/P$ . Since  $P$  is a nil ideal, we may assume from the beginning that  $e$  is in  $E$ . By hypothesis,  $eR(1-e) \cdot (1-e)Re = (1-e)Re \cdot eR(1-e)$ , and so  $eR(1-e)Re = 0$ . Hence  $\bar{e}\bar{R}(1-\bar{e})\bar{R}\bar{e} = 0$ . By the semiprimeness of  $\bar{R}$ , we get  $\bar{e}\bar{R}(1-\bar{e}) = 0$ , and therefore  $\bar{e}\bar{x} = \bar{e}\bar{x}\bar{e}$  for all  $x \in R$ . Furthermore,  $\bar{e}\bar{R}(1-\bar{e})\bar{R} = 0$  yields  $(1-\bar{e})\bar{R}\bar{e} = 0$ , and therefore  $\bar{x}\bar{e} = \bar{e}\bar{x}\bar{e}$  for all  $x \in R$ . Thus, we have seen that  $\bar{e}$  is central.

(2) Let  $J$  be the Jacobson radical of the  $\pi$ -regular ring  $R$ . Obviously,  $J/P$  is a nil ideal of  $R/P$ . Since  $R/J \simeq (R/P)/(J/P)$  and  $R/P$  is normal by (1), it is easy to see that  $R/J$  is normal and any nilpotent element of

$R/J$  generates a nil right ideal. Hence,  $R/J$  is reduced and  $N$  coincides with  $J$ . The reduced  $\pi$ -regular ring  $R/N$  is strongly regular (see, e.g. [2]).

*Proof of Theorem 2.* (1) As in the proof of Theorem 1, we may assume that  $p^a R = 0$  with some prime  $p$ . Then, by Lemma 1, there exists a positive integer  $m$  such that  $x^m = x^{2m}$  for all  $x \in R$ . Hence, by Lemma 2 (2),  $N$  coincides with the Jacobson radical of  $R$  and  $R/N$  satisfies the identity  $x - x^{m+1} = 0$ . By Jacobson's commutativity theorem,  $R/N$  is commutative.

(2) This is clear by (1).

(3) By the proof of (1) and [6, Theorem 1],  $R$  is commutative.

**Lemma 3.** *If  $R$  satisfies  $(i)_{2^k}$  and  $2^a R = 0$ , then  $N$  is a nil ideal and  $R/N$  is a Boolean ring.*

*Proof.* Let  $n = 2^k$ . Obviously,  $y = x + x^2 + \cdots + x^n \in N$ , and so  $x - x^{n+1} = y - xy \in N$ . Then, noting that  $x(1-x)^n - (x - x^{n+1}) \in 2R \subseteq N$ , we readily see that  $x - x^2 = x(1-x) \in N$ . Now, we claim that for any prime  $q$ ,  $(\text{GF}(q))_2$  fails to satisfy  $(i)'_n$ . If  $q \neq 2$  then  $x = E_{12}$  does not satisfy  $(i)'_n$ . On the other hand, if  $(\text{GF}(2))_2$  satisfies  $(i)'_n$  then, as we have seen just above,  $x - x^2$  is nilpotent for all  $x \in (\text{GF}(2))_2$ . But this is not true for  $x = E_{11} + E_{12} + E_{21}$ . Thus, by [3, Proposition 2],  $D$  is a nil ideal, and so  $R/N$  is a Boolean ring.

*Proof of Theorem 3.* By Remark 1 (3),  $hR = 0$  for some positive integer  $h = 2^a h'$ ,  $(2, h') = 1$ . Then,  $R = R_1 \oplus R_2$ , where  $h'R_1 = 0$  and  $2^a R_2 = 0$ . If  $a$  is an element of  $R_1$  with  $a^2 = 0$  then, by  $(i)_{2^k}$ , we can easily see that  $2^k a = 0$ , and therefore  $a = 0$ . This proves that  $R_1$  is a reduced ring and  $N \subseteq R_2$ . Now, the assertion is an easy combination of Theorem 2 (1) and Lemma 3.

*Proof of Theorem 4.* In view of Theorem 3, it remains only to prove that  $R_2$  is commutative. Obviously, every element of  $R_2$  is of the form  $e + a$ , where  $e \in E \cap R_2$  and  $a \in N$ . Since  $E$  is central and  $N$  is commutative, it is immediate that  $R_2$  is commutative.

**Remark 3.** As the following example shows, Theorem 4 is not true if we replace  $2^k$  by an arbitrary positive integer: Let

$$R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a^2 & 0 \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \text{GF}(4) \right\}.$$

Then  $12R = 0$  and  $R$  is a normal ring satisfying (i)<sub>12</sub> and (ii). But  $R$  is not commutative and  $R/N$  is not Boolean either.

*Proof of Theorem 5.* “Only if”: Obviously, (\*) implies (i)<sub>2</sub>. We claim that  $R$  is normal. To see this, given  $e \in E$  and  $x \in R$ , we put  $a = ex(1-e)$ . Since

$$\{(a-e) - (a-e)e\} \circ \{-e - e(a-e)\} = a \circ (-a) = -a^2 = 0,$$

(\*) shows that  $a - e = -e$ , and so  $a = 0$ , i.e.,  $ex = exe$ . A similar argument gives  $xe = exe$ , and hence  $ex = xe$ . It is easy to see that  $8R = 0$ , and therefore  $R$  is a normal, left  $s$ -unital ring with  $2^3R = 0$  and satisfies (i)<sub>2</sub>. We shall show that  $R$  satisfies (ii). Let  $x$  be an arbitrary element of  $R$ , and let  $e$  be such that  $ex = x$ . Since  $(x+x^2)^{(2)} = 0 = (-x+x^2)^{(2)}$  by (i)<sub>2</sub>, we readily obtain  $4(x+x^3) = 0$ , i.e.,  $4x = 4x^3$ . Replacing  $x$  by  $e+x$  in the last, we get

$$4x + 4x^2 = 4(e+x)x = 4(e+x)^3x = 4x + 4x^2 + 4x^3 - 4x^4.$$

Combining this with  $4x = 4x^3$ , we see that  $4x = 4x^4 = 4x^3x = 4x^2$ . Since  $x^4 = -2x^3 - 3x^2 - 2x$  by (i)<sub>2</sub>, we get

$$x^5 = -2x^4 - 3x^3 - 2x^2 = -2(-2x^3 - 3x^2 - 2x) - 3x^3 - 2x^2 = x^3 + 4x^2 - 4x = x^3.$$

This implies that  $a^3 = 0$  for all  $a \in N$ , and therefore

$$a + a^2 = (a + a^2)^{(2)} \circ (-a) = -a, \text{ i.e., } a^{(2)} = 0.$$

Noting that  $N$  is a nil ideal by Theorem 1, we get for any  $a, b \in N$

$$a \circ b = a \circ (a \circ b)^{(2)} \circ b = b \circ a,$$

which shows that  $N$  is commutative. Thus, by Theorem 4,  $R$  is commutative and  $R/N$  is a Boolean ring.

“If”: First, we claim that every quasi-regular element of  $R$  is nilpotent. In fact, if  $a$  is quasi-regular then the nilpotency of  $a + a^2$  yields that of  $a$ . Obviously,  $(x+x^2)^{(2)} = 0$  for all  $x \in R$ . Conversely, if  $(x+xy) \circ (y+yx) = 0$  then  $x+xy$  is nilpotent, and hence  $y+xy = (x+xy)^{(2)} \circ (y+yx) = x+xy$ , whence it follows that  $y = x$ .

**Remark 4.** In order to prove the only if part of Theorem 5, we quoted Theorems 1 and 4. In case  $R$  has 1, we can prove it more directly. In fact,  $x^5 = x^3$  yields that  $u^2 = 1$  for every unit  $u$  in  $R$ . If  $u, v$  are units in  $R$  then  $uv = (uv)^{-1} = v^{-1}u^{-1} = vu$ . In particular, if  $a$  and  $b$  are in  $N$  then  $[a, b] = [1+a, 1+b] = 0$ , and so  $N$  is commutative. Now, let  $a \in N$ ,  $x \in R$ . Since  $(x+x^2)^6 = -8(x+x^2)^3 = 0$  and  $(x^2+x^4)^6 = 0$  by (i)<sub>2</sub> and  $x^4$

is a central idempotent, we have  $[a, x] = [a, x - x^4] = [a, x + x^2] - [a, x^2 + x^4] = 0$ , and hence  $N \subseteq C$ . Therefore,  $x = (x + x^2) - (x^2 + x^4) + x^4 \in C$  for all  $x$  in  $R$ , and hence  $R$  is commutative.

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