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ON THE NILPOTENCY INDEX OF THE RADICAL OF A GROUP ALGEBRA. IV

Dedicated to Professor Kentaro Murata on his 60th birthday

KAORU MOTOSE

Throughout this paper, we shall use the following notations: Let p be a fixed prime number, let G be a finite group with a p -Sylow subgroup P of order p^a , let KG be a group algebra of G over a field K of characteristic p and let $t(G)$ be the nilpotency index of the radical $J(KG)$ of KG . Further, given a finite subset S of KG , $\hat{S}$ denotes the sum of all elements of S .

First, in § 1, we shall investigate $t(G)$ of a group G with a split (B, N) -pair. Next, in § 2, we shall state some remarks concerning a theorem of Morita [5]. § 3 is devoted to studying a p -solvable group G with $t(G) = \alpha(p-1)+1$. Unfortunately, we have not characterized yet a group G of minimal order such that G is p -solvable, $t(G) = \alpha(p-1)+1$ and P is not elementary abelian. However, we can establish the structure of such a group under certain extra conditions (Theorem 12). § 4 contains an example of a group H satisfying the following conditions:

(1) H possesses all the conclusions of Lemma 11.

(2) $t(H) > \beta(p-1)+1$ where p^β is the order of a p -Sylow subgroup of H .

Finally, by making use of results in § 3, we shall prove that there exist no finite rings satisfying certain conditions (§ 5).

1. We begin with stating a lemma which is efficient in studying the nilpotency index of the radical of a group algebra.

Lemma 1. *Let A be a ring. Let B , I and J be subsets of A satisfying the following conditions:*

(1) $IAI = IBI$, (2) $IJ = JI$, (3) $BJ \subseteq JB$.

Then $(JIA)^n \subseteq J^n IA$. Moreover, if $J^n = 0$ then JIA is contained in the radical of A .

Proof. Clearly, the result holds for $n = 1$. Assume the result for n . Then we conclude that $(JIA)^{n+1} \subseteq J^n IAJIA = J^n IAIJA = J^n IBIJA = J^n IBJIA \subseteq J^n IJBIA = J^{n+1} IBIA \subseteq J^{n+1} IA$.

Theorem 2. *Suppose that a group G has finite subgroups H and U such that $G = UN_G(H)U$ and $H \subseteq N_G(U)$. Then $(J(KH)\widehat{U}KG)^{t(H)} = 0$ and $J(KH)\widehat{U} \subseteq J(KG)$.*

Proof. Taking A to be KG , B to be $KN_G(H)$, I to be $\{\widehat{U}\}$, and J to be $J(KH)$ in Lemma 1, we readily obtain the conclusion.

We suggest the effectiveness of Theorem 2 by giving the following example which played a fundamental role in [8].

Example 3. Let $q = p^e$, r a divisor of $q-1$, and s a multiple of p with $(s, q-1) = 1$. Let H be the Galois group of $R = GF(q^s)$ over $S = GF(q)$ and let T be $\langle b^r \rangle$ where b is a generator of the multiplicative group of R . We consider the following permutation groups on R : $V = \{v_a: x \rightarrow x+a \mid a \in R\}$ and $U = \{u_t: x \rightarrow tx \mid t \in T\}$, and put $G = \langle H, U, V \rangle$. Since $q-1$ and $(q^s-1)/(q-1)$ are relatively prime, every element of R is a product of an element of T and an element of S . This shows that $G = UC_G(H)U$ since $C_G(H)$ contains $\{v_a \mid a \in S\}$ and $u_tv_au_t^{-1} = v_{ta}$. Noting that $hu_th^{-1} = u_{h(t)}$ for all $h \in H$, by Theorem 2, we obtain that $J(KH)\widehat{U} \subseteq J(KG)$ and the nilpotency index of $J(KH)\widehat{U}KG$ is the p -part of s .

Theorem 2 is also applicable to finite groups of Lie type (see [1]).

Proposition 4. *Suppose that a finite group G has a split (B, N) -pair of characteristic r such that B is a semi-direct product of a normal r -subgroup U and an abelian r' -subgroup $H = B \cap N$ (see [1]). Then $J(KH)\widehat{U} \subseteq J(KG)$.*

Proof. Since $G = UNU$ by Bruhat decomposition and H is a normal subgroup of N , the assertion follows from Theorem 2.

Although the next is stated for a special type of a finite group, similar observation is possible for finite groups of Lie type, too.

Proposition 5. *Let r be a prime number, $q = r^e$, p an odd prime divisor of $q-1$, and $G = SL(2, q)$. Then $t(G)$ is the p -part p^m of $q-1$.*

Proof. Obviously, G has a split (B, N) -pair, where B is the subgroup of upper triangular matrices and N is the subgroup of monomial matrices. Thus $H = B \cap N$ is the subgroup of diagonal matrices which is isomorphic to the multiplicative group of the finite field of q elements. Thus, by

Proposition 4, we have $p^m = t(H) \leq t(G)$. Since the order of G is $q(q-1) \cdot (q+1)$, a p -Sylow subgroup of G is a cyclic group of order p^m , and so $t(G) = p^m$ by Dade's theorem [2] (see also [7]).

2. Throughout this section, we shall use the following notations: Let N be a normal subgroup (not necessarily a p' -subgroup) of a finite group G and let e be a centrally primitive idempotent of KN . We set $H = \{x \in G \mid xex^{-1} = e\}$ and $\tilde{e} = \sum_{i=1}^s a_i ea_i^{-1}$, where $\{a_i\}$ ($a_1 = 1$) is a set of representatives of the right cosets of H in G .

The following interesting theorem has been proved by K. Morita.

Theorem 6 (Morita [5]). *If K is algebraically closed and KNe is simple, then $KG\tilde{e}$ is isomorphic to a complete matrix algebra over a twisted group algebra of H/N .*

The next is immediate from the preceding theorem.

Corollary 7. *Assume that K is algebraically closed. If $N = H$ and KNe is simple, then $KG\tilde{e}$ is simple.*

This result applies especially to the radical of a group algebra of a Frobenius group.

Corollary 8 ([6]). *Let G be a Frobenius group with kernel N and complement W . If p divides the order of W then $J(KG) = J(KW)\hat{N}$.*

Proof. We may assume, in the usual way, that K is algebraically closed. If e is not equal to $|N|^{-1}\hat{N}$, then $H = N$ implies that $KG\tilde{e}$ is simple (see [3, the proof of (25.4)]). Thus we obtain the assertion.

As an application of Lemma 1, we have the following

Proposition 9. $J(KHe) = J(KH)e \subseteq J(KG)$.

Proof. Setting $A = KG$, $B = KH$, $I = \{e\}$ and $J = J(KH)$, we have $IAI = eKGe = \sum_{i=1}^s KHea_i ea_i^{-1} a_i = KHe = eKHe = IBI$. Also, it is easy to check other assumptions of Lemma 1. Thus we have $J(KH)e \subseteq J(KG)$.

The next shows that the converse of Corollary 7 holds.

Corollary 10. *If $N = H$ and $KG\tilde{e}$ is simple, then KNe is simple.*

Proof. By Proposition 9, $0 = J(KG)\bar{e} \supseteq J(KN)e\bar{e} = J(KN)e = J(KNe)$. Thus KNe is simple.

3. Throughout this section, let G be a group of the minimal order which satisfies the following conditions:

1. G is a p -solvable group.
2. P is not elementary abelian.
3. $t(G) = \alpha(p-1)+1$.

We show that G possesses the properties listed in the following lemma.

Lemma 11. (1) $O_{p'}(G) = 1$.

(2) $U = O_p(G)$ ($\neq 1$) is elementary abelian and $G = O_{p,p',p}(G)$.

(3) $|G| \leq p^{p+1}(p^p-1)/(p-1)$.

(4) $U = [U, V]$, $C_U(V) = 1$ and G is a semi-direct product of U by $N_G(V)$ where V is a p' -subgroup such that $O_{p,p'}(G) = UV$.

(5) $V = [W, V]$ where W is a p -Sylow subgroup of $N_G(V)$.

(6) $N_G(V) = WV$ is isomorphic to a subgroup of $\text{Aut}(U) = \text{GL}(U)$.

(7) U is a minimal normal subgroup of UV .

Proof. (1) We have $t(G) \geq t(G/O_{p'}(G)) \geq \alpha(p-1)+1$ (see [12]). Thus $O_{p'}(G) = 1$ by the minimality of the order of G .

(2) From the inequality $t(G) \geq t(G/U) + t(U) - 1$ (see [12]) we see that both U and P/U are elementary abelian. Hence G/U is of p -length 1, and so $H = O_{p,p',p}(G)$ is a normal subgroup of G whose index is a p' -number. Since $t(G) = t(H)$ (see [11]), we get $G = H = O_{p,p',p}(G)$.

(3) It is known that there exists a group H of order $p^{p+1}(p^p-1)/(p-1)$ such that $t(H) = p^2$ and a p -Sylow subgroup of H is nonabelian (see [8]). Hence, we have $p^{p+1}(p^p-1)/(p-1) \geq |G|$.

(4) We can see that $G = N_G(V)U$ by Frattini argument, and $N_U(V) = C_U(V)$. Thus, to prove the last assertion, it suffices to show that $C_U(V)$ is trivial. In view of $C_G(U) = U$ (see [4, Theorem 3.2, p.228]), V can be regarded as an automorphism group of U by conjugation, and consequently $U = C_U(V) \times [V, U]$ (see [4, Theorem 2.3, p.177]). Clearly, $[V, U] = [UV, U]$ is a non-trivial normal subgroup by $O_{p'}(G) = 1$. If $C_U(V)$ is non-trivial, then P must be elementary abelian since $C_U(V) = C_U(UV)$ is normal and so G can be embedded in $(G/C_U(V)) \times (G/[V, U])$. This contradiction shows that $C_U(V)$ is trivial.

(5) Set $V_1 = [V, W]$. Then $H = WV_1U$ is a normal subgroup of G whose index is a p' -number. Since $t(G) = t(H)$ (see [11]) and $P = WU$,

we have $G = H$ and so $V = [V, W]$.

(6) The assertion follows from $C_G(U) = U$.

(7) Let U_1 be a minimal normal subgroup of UV contained in U . We prove first that U_1 is normal in G . Assume that $U_1^\sigma \neq U_1$ for some $\sigma \in W$. Then $U_1^\sigma \cap U_1 = 1$ since $U_1^\sigma \cap U_1$ is normal in UV . Noting that V is a p' -group and U is elementary abelian, we can easily see that there exists a normal subgroup U_2 of UV such that $U = U_1 \times U_1^\sigma \times U_2$. Since $J(KG)$ contains the kernel of the natural homomorphism $KG \rightarrow KG/U$, we obtain $J(KG) \supseteq J(KW)\hat{V} + J(KU)KG$ (see [8]). We set $U_3 = U_1^\sigma \times U_2$. Then, in view of $t(G) = \alpha(p-1)+1$, we have $0 = \hat{W}\hat{V}\hat{U}_1(1-\sigma)\hat{V}\hat{U}_3 = \hat{W}\hat{V}(\hat{U}_1 - \hat{U}_1^\sigma)\hat{V}\hat{U}_3 = |V|\hat{W}\hat{V}(\hat{U}_1 - \hat{U}_1^\sigma)\hat{U}_3 = |V|\hat{W}\hat{V}\hat{U}_1\hat{U}_3 = |V|\hat{G}$, which is impossible. Hence U_1 is normal in G . Since V is a p' -group, we have $U = U_1 \times U_2 \times \cdots \times U_t$ where every U_i is a minimal normal subgroup of UV . Then, by the preceding argument, every U_i is a normal subgroup of G . If $t \geq 2$, then P must be elementary abelian since G can be embedded in $G/U_1 \times G/U_2 \times \cdots \times G/U_t$. This contradiction shows that $t = 1$ and so U is a minimal normal subgroup of G .

In the remainder of this section, we preserve the notations used in Lemma 11. The next is the main result of this section.

Theorem 12. *If V is abelian, then G can be regarded as a permutation group on $GF(p^p)$ such that $U = \{x \rightarrow x+b \mid b \in GF(p^p)\}$, W is the Galois group of $GF(p^p)$ over $GF(p)$ and that $V \subseteq \{x \rightarrow tx \mid t \in \langle \lambda^{p-1} \rangle\}$ where λ is a generator of the multiplicative group of $GF(p^p)$.*

Proof. By virtue of Lemma 11 (7) and the proofs of [9, Proposition 19.8 and Theorem 19.11], we can regard G as a permutation group on some $GF(p^n)$ such that $U = \{x \rightarrow x+b \mid b \in GF(p^n)\}$, W is a subgroup of the Galois group of $GF(p^n)$ over $GF(p)$ and that V is a subgroup of $\{x \rightarrow ax \mid a \in GF(p^n)^*\}$. Since W is elementary abelian and cyclic, it follows that W is of order p . Thus $n = pr$ with some integer r , and $p^{2p+1} > p^{p+1}(p^p-1)/(p-1) \geq |G| > |W| |U| = p^{rp+1}$, which implies $r = 1$. Now, by Lemma 11 (5), we can easily see that the order of V divides $(p^p-1)/(p-1)$. This completes the proof.

In [10], Y. Tsushima stated that if H is a p -solvable group with a regular p -Sylow subgroup S of order p^p and $t(H) = \beta(p-1)+1$, then S is elementary abelian. On page 37, line 11 [10], he claimed that since P is of exponent p , G is of p -length 1 by Hall Higman's theorem. However,

unfortunately, Tsushima's argument is unjustifiable for Fermat primes. The next shows that Tsushima's result holds under extra assumption that $O_{p',p,p'}(H)/O_{p',p}(H)$ is abelian.

Corollary 13. *Let H be a p -solvable group with a p -Sylow subgroup S of order p^β and $O_{p',p,p'}(H)/O_{p',p}(H)$ abelian. If $t(H) = \beta(p-1)+1$ and S is regular, then S is elementary abelian.*

Proof. Let H be a counter example of the minimal order. Then S is not regular by Theorem 12 and [8, Lemma 4]. Hence S is elementary abelian.

4. In this section, for $p=3$, we shall give an example of a group H which possesses all the properties listed in Lemma 11 but satisfies $t(H) > \beta(p-1)+1$ where p^β is the order of a p -Sylow subgroup of H .

We set $a = \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$, $b = \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$, and $c = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ in $M = SL(2, 3)$. Then $a^4 = 1$, $b^2 = a^2$, $b^a = a^{-1}ba = b^{-1}$, $c^3 = 1$, $a^c = a^3b$, $b^c = a$, and $Q = \langle a, b \rangle$ is a normal 3'-subgroup of M . Let H be a semi-direct product of $U = \langle x, y \mid x^3 = y^3 = 1, xy = yx \rangle$ by M with respect to the identity map of M . Then we have the following relations:

$$x^a = y^2, y^a = x, x^b = xy^2, y^b = x^2y^2, x^c = x, y^c = xy.$$

It is easy to see that H possesses all the properties listed in Lemma 11. We set $a \circ b = a + b - ab$, $\chi = 1 + x + x^2$, $\nu = 1 + y + y^2$, $f = a^2 - 1$ and $\tau = c(1 + a \circ b)f$ in the group algebra KH . Then we have the following

Lemma 14. (1) f is a central idempotent in KM .

(2) $a^2f = -f$, $fyf = fy^2f$, and $f\chi = \chi f$.

(3) τ is central in KM . In particular, c commutes with $(a \circ b)f$.

(4) $\tau^2 = c^2(1 - a \circ b)f$ and $\tau^3 = f$.

(5) $J(KH) \supseteq J(KT)$ where KT is the group algebra of a cyclic group $T = \{f, \tau, \tau^2\}$ over K .

Proof. (1)–(4) can be proved by direct verification.

(5) It follows from (3) that KT is contained in the center of KM . Since $J(KH)$ contains the kernel of the natural homomorphism $KH \rightarrow KH/U = KM$, we obtain $J(KH) \supseteq J(KM) \supseteq J(KT)$.

Lemma 15. $\chi f(y + \tau y \tau^2 + \tau^2 y \tau) f = -\bar{U} f(1 + b + ab)$.

Proof. By making use of Lemma 14, we obtain the following equations :

$$\begin{aligned}
 \chi f \tau y \tau^2 f &= \chi c(1+a \circ b) f y c^2(1-a \circ b) f \\
 &= \chi(1+a \circ b) f c^{-2} y c^2(1-a \circ b) f \\
 &= \chi f(1-b^{-1} \circ a^{-1}) x^2 y f(1-a \circ b) \\
 &= \chi f(x^2 y + (x^2 y)^a a + (x^2 y)^b b - (x^2 y)^{ab} ab) f(1-a \circ b) \\
 &= f \chi(x^2 y + x y a + x b - y a b) f(1-a \circ b) \\
 &= \chi f(y + y a + b - y a b)(1-a-b+ab) f \\
 &= \chi f(1+a+b+ab-ya-yb-yab) f. \\
 \chi f \tau^2 y \tau f &= \chi c^2(1-a \circ b) f y c(1+a \circ b) f \\
 &= \chi(1-a \circ b) f y c(1+a \circ b) f \\
 &= \chi f(1+b^{-1} \circ a^{-1}) x y f(1+a \circ b) \\
 &= \chi f(x y - (x y)^a a - (x y)^b b + (x y)^{ab} ab) f(1+a \circ b) \\
 &= f \chi(x y - x y^2 a - y b + x^2 ab) f(1+a \circ b) \\
 &= \chi f(y - y^2 a - y b + ab) f(1+a \circ b) \\
 &= \chi f(y - y a - y b + ab)(1+a+b-ab) f \\
 &= \chi f(1-a+b+ab+ya-yb-yab) f.
 \end{aligned}$$

Thus, from those above we get

$$\begin{aligned}
 \chi f(y + \tau y \tau^2 + \tau^2 y \tau) f &= \chi f(y-1)(1+b+ab) f \\
 &= -\chi f y f(1+b+ab) = -\widehat{U} f(1+b+ab).
 \end{aligned}$$

We are now ready to establish $t(H) > 7 = 3(3-1)+1$.

Proposition 16. $t(H) \geq 9$.

Proof. It follows from Lemma 14 (5) that $J(KH)^8$ contains an element $\chi \widehat{T} \nu \widehat{T}$ where $\widehat{T} = f + \tau + \tau^2$. By Lemma 15, we have

$$\begin{aligned}
 \chi \widehat{T} \nu \widehat{T} &= \chi \widehat{T} (f y f + f y^2 f) \widehat{T} = -\chi \widehat{T} f y f \widehat{T} = -\chi f \widehat{T} y \widehat{T} f \\
 &= -\chi f(y + \tau y \tau^2 + \tau^2 y \tau) \widehat{T} f = \widehat{U} f(1+b+ab) \widehat{T} \\
 &= \widehat{U} \widehat{T}(1+b+ab) = \widehat{U}(1+c+c^2)(1+b+ab) f \neq 0.
 \end{aligned}$$

This completes the proof.

5. In this section, we shall prove that there exist no finite rings R satisfying the following conditions:

- 1) R is a finite ring of characteristic p .
- 2) R admits an automorphism σ of order p .
- 3) $\text{Tr}(a) = 0$ for every $a \in R$ where $\text{Tr}(a)$ means the $\langle \sigma \rangle$ -trace of a .
- 4) There exists a p' -subgroup T of the unit group of R such that T

is abelian, $\sigma(T) \subseteq T$, $T \cap R^\sigma = 1$ and $R = \{tc \mid t \in T, c \in R^\sigma\}$, where $R^\sigma = \{c \in R \mid \sigma(c) = c\}$.

Suppose to the contrary that there exists such a ring, and consider a permutation group $H = \langle U, V, W \rangle$ on R (acting on the left), where $U = \{u_r : x \rightarrow x+r \mid r \in R\}$, $V = \{v_t : x \rightarrow tx \mid t \in T\}$ and $W = \langle \sigma \rangle$. Since the condition $T \cap R^\sigma = 1$ implies that WV is a Frobenius group, we get $J(KH) = J(KW)\hat{V}KH + J(KU)KH$ (see Corollary 8 and [8, Proposition 3]). On the other hand, $R = \{tc \mid t \in T, c \in R^\sigma\}$ gives $H = VC_G(W)V$ (see Example 3). Thus $(J(KW)\hat{V}KH)^p = 0$ by Theorem 2 and hence $t(G) = \beta(p-1)+1$ where $|U| = p^{\beta-1}$. The condition 3) implies that $(\sigma^h u_a)^p = \sigma^{hp} u_{\text{Tr}(a)} = 1$, and so $S = WU$ is of exponent p . Hence a p -Sylow subgroup S of H is regular. Then, since V is abelian, it follows from Corollary 13 that S is elementary abelian, which contradicts the fact that S is not abelian.

REFERENCES

- [1] C.W. CURTIS: Modular representations of finite groups with split (B, N) -pairs, Lecture Notes in Math. 131, Springer, Berlin-Heidelberg-New York, 1970, 57—95.
- [2] E.C. DADE: Blocks with cyclic defect groups, Ann. of Math. 84 (1966), 20—48.
- [3] W. FEIT: Characters of Finite Groups, Benjamin, New York, 1967.
- [4] D. GORENSTEIN: Finite Groups, Harper & Row, New York-Evanston-London, 1967.
- [5] K. MORITA: On group rings over a modular field which possess radicals expressible as principal ideals, Sci. Rep. Tokyo Bunrika Daigaku 4 (1951), 177—194.
- [6] K. MOTOSE: On radicals of group rings of Frobenius groups, Hokkaido Math. J. 3 (1974), 23—34.
- [7] K. MOTOSE: On radicals of principal blocks, Hokkaido Math. J. 6 (1977), 255—259.
- [8] K. MOTOSE: On the nilpotency index of the radical of a group algebra, III, J. London Math. Soc. 25 (1982), 39—42.
- [9] D.S. PASSMAN: Permutation Groups, Benjamin, New York, 1968.
- [10] Y. TSUSHIMA: Some notes on the radical of a finite group ring II, Osaka J. Math. 16 (1979), 35—38.
- [11] O.E. VILLAMAYOR: On the semi-simplicity of group algebras, II, Proc. Amer. Math. Soc. 10 (1959), 27—31.
- [12] D.A.R. WALLACE: Lower bounds for the radical of the group algebra of a finite p -soluble group, Proc. Edinburgh Math. Soc. 16 (1968/69), 127—134.

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