

Mathematical Journal of Okayama University

Volume 25, Issue 2

1983

Article 5

DECEMBER 1983

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ON A THEOREM OF MAYNE

YASUYUKI HIRANO, ARIF KAYA and HISAO TOMINAGA

Throughout, R will represent an (associative) ring with center C . Let S be a subset of R . An (additive group) endomorphism T of R is said to be *centralizing* (resp. *skew-centralizing*) on S if $[s^T, s] = s^T s - s s^T \in C$ (resp. $(s^T, s) = s^T s + s s^T \in C$) for every $s \in S$. More generally, T is defined to be *semicentralizing* on S if $[s^T, s] \in C$ or $(s^T, s) \in C$ for every $s \in S$. In case $S = R$, we say simply T is *centralizing* (resp. *skew-centralizing*) or *semicentralizing* according as so is T on R .

Recently, in [5], by making use of his previous result in [4], J.H. Mayne proved that if a prime ring R has a nontrivial ring automorphism T and a nonzero ideal U such that T is centralizing on U and $U^T \subseteq U$ then R is commutative. However, his proof is based on an unjustifiable assertion that T induces a ring automorphism on U . Incidentally, it should be mentioned that the result of [4] itself was claimed (even for a nontrivial surjective ring endomorphism) back in 1959 by M.F. Smiley [7, Remark 2].

In the present paper, we shall prove the following theorem which greatly generalizes [5, Theorem] and includes [1, Lemma] and [3, Corollary], as well.

Theorem 1. *Let U be a nonzero ideal of a prime ring R .*

(1) *Let T be a nontrivial ring endomorphism of R ($T \neq 1_R$). If T is semicentralizing on U , U^T is an ideal of R and $(U \cap U^T)^T$ is nonzero, then R is commutative.*

(2) *Let T be a nontrivial derivation of R ($T \neq 0_R$). If T is centralizing (resp. skew-centralizing) on U , then R is commutative.*

In preparation for proving our theorem, we state first several lemmas.

Lemma 1. *Let R be a prime ring, and I a right ideal of R .*

(1) *If I is nonzero and commutative, then R is commutative.*

(2) *Let T be a ring endomorphism of R . If I is nonzero and T is trivial on I , then T is itself trivial.*

(3) *Let T be a derivation of R . If I is nonzero and T is trivial on I , then T is itself trivial.*

(4) *Let T be a nontrivial derivation of R , and x an element of R . If $xR^T = 0$ then $x = 0$.*

(5) If there exists a positive integer n such that $x^n=0$ for all $x \in I$, then $I=0$.

Proof. (2), (3) and (4) are respectively [5, Lemmas 3, 2] and [6, Lemma 1] with routine proofs. (1) is [5, Lemma 4] and (5) is immediate by [2, Lemma 1.1]. However, for the sake of self-containedness, we prove (1) and (5).

(1) Given $a, b \in I$ and $x, y \in R$, we have $ab[x, y] = abxy - bayx = bxa y - aybx = 0$, namely $I^2[x, y] = 0$. Hence, $[x, y] = 0$ for all $x, y \in R$.

(5) We proceed by induction on n . First, we claim that $aI=0$ for any $a \in I$ with $a^2=0$. Let $A=aI$ and $S=\{x \in A \mid xA=0\}$. As is easily seen, S is a prime ideal of A . Furthermore, since $(ay)^{n-1}aI = (a+ay)^nI = 0$ for any $y \in R$, we see that $x^{n-1} \in S$ for all $x \in A$. Hence, by induction hypothesis, $A/S=0$, i.e., $A^2=0$, whence it follows that $aI=0$. Now, let $W=\{x \in I \mid xI=0\}$. Then W is a prime ideal of I . Since the above claim tells us that $x^{n-1} \in W$ for all $x \in I$, our induction hypothesis shows $I=W$, i.e., $I^2=0$. Hence we have $I=0$.

Lemma 2. Let $T: x \rightarrow x'$ be an endomorphism of R , and U an additive subgroup of R . Let $[U] = \{u \in U \mid [u', u] \in C\}$ and $(U) = \{u \in U \mid (u', u) \in C\}$.

(1) Let $u, v \in [U]$ (resp. (U)). Then $u+v \in [U]$ (resp. (U)) if and only if $u-v \in [U]$ (resp. (U)).

(2) If $v \in (U)$, then $[v', v^2] = [v, v'^2] = 0$.

Proof. (1) follows from $[u'-v', u-v] = -[u'+v', u+v] + 2([u', u] + [v', v])$ (resp. $(u'-v', u-v) = -(u'+v', u+v) + 2((u', u) + (v', v))$), and (2) is obvious by $[x, y^2] = [(x, y), y]$.

Lemma 3. Let $T: x \rightarrow x'$ be a ring endomorphism of a prime ring R of characteristic not 2 which is semicentralizing on a nonzero ideal U , and let $[U], (U)$ be as in Lemma 2.

(1) If $v \in U \setminus [U]$, then $v^2v'^2 = v'^2v^2 = 0$.

(2) If U^T is a nonzero ideal of R and $[U] \neq U$, then there is no positive integer n such that $v^n=0$ for all $v \in U \setminus [U]$.

Proof. (1) By Lemma 2 (2), we have

$$[v'^2 + v', v^2 + v] = [v'^2 - v', v^2 - v] = [v', v] \notin C,$$

which means that $v^2 + v \notin [U]$ and $v^2 - v \notin [U]$. Then, by Lemma 2 (1), $(v^2 + v) - (v^2 - v) = 2v \in [U]$ shows that $2v^2 = (v^2 + v) + (v^2 - v) \in (U)$, and

so $v^2 \in (U)$. Hence, by Lemma 2 (2), $2v'^2v^2 = (v'^2, v^2) \in C$, i.e., $v'^2v^2 \in C$. Furthermore, again by Lemma 2 (2),

$$0 = v'^2[v^2 + v', (v^2 + v)^2] = 2v'^2[v', v^3] = 2v'^2v^2[v', v],$$

i.e., $v'^2v^2[v', v] = 0$. Since $v'^2v^2 \in C$ and R is prime, $[v', v] \neq 0$ implies that $v^2v'^2 = v'^2v^2 = 0$.

(2) Suppose that $v'^n = 0$ for all $v \in U \setminus [U]$. We shall show $v' = 0$, which contradicts $v \notin [U]$. In order to see this, it suffices to show that $v'^{n-1} = 0$ (if $n > 1$). Let u be an arbitrary element of U . If $uv^{n-1} \in [U]$ then $(u'v'^{n-1})^n = 0$ by assumption. Next, suppose that $uv^{n-1} \in [U]$. Since

$$(uv^{n-1} + v) + (uv^{n-1} - v) = 2uv^{n-1} \in [U]$$

and

$$(uv^{n-1} + v) - (uv^{n-1} - v) = 2v \notin [U],$$

we see that either $uv^{n-1} + v \notin [U]$ or $uv^{n-1} - v \notin [U]$ (Lemma 2 (1)). Hence, either

$$(u'v'^{n-1})^{n+1} = u'v'^{n-1}(u'v'^{n-1} + v')^n = 0$$

or

$$(u'v'^{n-1})^{n+1} = u'v'^{n-1}(u'v'^{n-1} - v')^n = 0.$$

We have therefore seen that $(u'v'^{n-1})^{n+1} = 0$ for all $u \in U$ and $v \in U \setminus [U]$. Since $U^T v'^{n-1}$ is a nil left ideal of bounded index, we get $v'^{n-1} = 0$ by Lemma 1 (5).

Corollary 1 (cf. [3, Theorem]). *Let T be a ring endomorphism of a prime ring R which is semicentralizing on a nonzero ideal U . If U^T is a nonzero ideal of R , then T is centralizing on U .*

Proof. We keep the notations in Lemma 2. If R is of characteristic 2, then $[x', x] = (x', x)$, and therefore T is centralizing on U . So, we assume henceforth that R is of characteristic not 2. Suppose $[U] \neq U$, and choose arbitrary $v \in U \setminus [U]$. Given $u \in U$, by making use of Lemma 2 (1) we can easily see that

$$uv^2u'v'^4 + v^2u'v'^4 = [uv^2 + v^2, (uv^2 + v^2)']v'^2 = (uv^2 + v^2, (uv^2 + v^2)')v'^2.$$

Hence,

$$uv^2u'v'^4 + v^2u'v'^4 = c_1v'^2 \text{ with some } c_1 \in C.$$

Similarly, considering $uv^2 - v^2$ instead of $uv^2 + v^2$, we get

$$uv^2u'v'^4 - v^2u'v'^4 = c_2v'^2 \text{ with some } c_2 \in C.$$

From those above, we obtain $2v^2u'v'^4 = (c_1 - c_2)v'^2$, and hence $2v^4u'v'^4 = 0$ again by Lemma 3 (1). This proves $v^4U^Tv'^4 = 0$, whence it follows that $v'^4 = 0$. But, this is impossible by Lemma 3 (2). We have thus proved that $[U] = U$.

Lemma 4. *Let $T : x \rightarrow x'$ be a derivation of a prime ring R of characteristic not 2 which is semicentralizing on a nonzero ideal U , and let $[U], (U)$ be as in Lemma 2.*

- (1) *If $v \in U \setminus [U]$, then $(v^2)' = 0$ and $v^2v' = v'v^2 = 0$.*
- (2) *If $C \cap U = 0$ and $v \in U \setminus [U]$, then $v'^3 = 0$ and $v^2 \neq 0$.*
- (3) *If $C \cap U$ is nonzero, then T is centralizing on U .*

Proof. (1) Since $(v^2)' = (v', v) \in C$ and $[v', v^2] = 0$ by Lemma 2 (2), we have

$$[(v^2 + v)', v^2 + v] = [(v^2 - v)', v^2 - v] = [v', v] \notin C,$$

which means that $v^2 + v \notin [U]$ and $v^2 - v \notin [U]$. Then, by Lemma 2 (1), $(v^2 + v) - (v^2 - v) = 2v \notin [U]$ shows that $2v^2 = (v^2 + v) - (v^2 - v) \in (U)$, and so $v^2 \in (U)$. Hence, $2(v^2)'v^2 = ((v^2)', v^2) \in C$, i.e., $(v^2)'v^2 \in C$. Furthermore, by Lemma 2 (2),

$$0 = (v^2)'[(v^2 + v)', (v^2 + v)^2] = 2(v^2)'[v', v^3] = 2(v^2)'v^2[v', v],$$

i.e., $(v^2)'v^2[v', v] = 0$. Since $(v^2)'v^2 \in C$ and R is prime, $[v', v] \neq 0$ implies $(v^2)'v^2 = 0$. Recalling here that $(v^2)' \in C$, we get $(v^2)' = 0$. Since $v^2 + v \notin [U]$, we have also $0 = ((v^2 + v)^2)' = ((v^2 + v)', v^2 + v) = (v', v^2 + v) = 2v'v^2$, and so $v'v^2 = v^2v' = 0$ by Lemma 2 (2).

(2) Observe that $vv' = -v'v$ and $uu' = \pm u'u$ for every $u \in U$. We prove first that $v^2 \neq 0$. In fact, if $v^2 = 0$ then for any $x \in R$ we have

$$vxv'v + xv'v'v = \{(v + xv)(v + xv)' \pm (v + xv)'(v + xv)\}v' = 0.$$

Replace x by $-x$ in the above to get $-vxv'v + xv'v'v = 0$. Hence $vRv'v = 0$, and therefore $v'v = 0$. But this contradicts $v \notin [U]$.

Next, we claim that $vv'^2 = 0$. Noting that $v^2v' = 0$ by (1), for any $x \in R$ we have

$$-v^2xvv'^2 - vxv^2xvv'^2 = \{(v + vxv)(v + vxv)' \pm (v + vxv)'(v + vxv)\}vv' = 0,$$

and similarly $v^2xvv'^2 - vxv^2xvv'^2 = 0$. Hence $v^2Rvv'^2 = 0$, and therefore $vv'^2 = 0$ by $v^2 \neq 0$.

Now, for any $x \in R$ we have

$$v xv'^3 + xv xv'^3 = \{(v + xv)(v + xv)' \pm (v + xv)'(v + xv)\} v'^2 = 0,$$

and similarly $-v xv'^3 + xv xv'^3 = 0$. Hence $v R v'^3 = 0$, and therefore $v'^3 = 0$.

(3) Suppose U contains an element v not contained in $[U]$. Choose an arbitrary nonzero $c \in C \cap U$. Because $c' \in C$, we have $[v' + c', v + c] = [v', v] \in C$, and so $v + c \in [U]$. Then, by (1),

$$0 = [((v + c)^2)', v] = [2cv' + 2c'v + (c^2)', v] = 2c[v', v],$$

i.e., $[v', v] = 0$. This contradiction proves that $[U] = U$.

Corollary 2. *Let $T : x \rightarrow x'$ be a derivation of a prime ring R , and U a nonzero ideal of R .*

(1) *If T is skew-centralizing on U , then it is centralizing on U .*

(2) *If T is semicentralizing on U and U^T is a left (resp. right) ideal of R , then T is centralizing on U .*

Proof. We may assume that $T \neq 0_R$ and R is of characteristic not 2.

(1) According to Lemma 4 (3), it suffices to show that $C \cap U$ is nonzero. Suppose, to the contrary, that $C \cap U = 0$. Then, for any $u \in U$ and $x \in R$,

$$(u^2x + uxu)' = \{(u + ux)^2 - u^2 - (ux)^2\}' = 0$$

and

$$(xu^2 + uxu)' = \{(u + xu)^2 - u^2 - (xu)^2\}' = 0.$$

From those above, we readily obtain $[x, u^2]' = 0$. This means that $DT = 0$, where D is the inner derivation of R effected by u^2 . Now, suppose that D is nontrivial. Let a, b and c be arbitrary elements of R . Obviously,

$$(*) \quad a'b^D + a^D b' = (ab)^{DT} = 0.$$

Noting that $b^{D^2}c' = (b^Dc)^{DT} = 0$ and $b^Dc' = -b'c^D$, we have

$$0 = (ab^D)'c^D + (ab^D)^Dc' = a'b^Dc^D + a^Db^Dc' = (a'b^D - a^Db')c^D,$$

namely $(a'b^D - a^Db')R^D = 0$. Hence, $a'b^D - a^Db' = 0$ by Lemma 1 (4). Combining this with (*), we get $a'R^D = 0$. Again by Lemma 1 (4), $a' = 0$ for all $a \in R$, i.e., $T = 0$. This contradiction proves $D = 0$, which tells us that $u^2 = 0$ for all $u \in U$. But, this is impossible by Lemma 1 (5).

(2) Suppose, to the contrary, that U contains an element v not contained in $[U]$. In view of Lemma 4 (3), it suffices to consider the case that $C \cap U = 0$. Let u be an arbitrary element of U . If $uv^2 \notin [U]$ then $(u'v^2)^3 = (uv^2)^3 = 0$ by Lemma 4. On the other hand, if $uv^2 \in [U]$ then it

is easy to see that either $v + uv^2 \in [U]$ or $v - uv^2 \in [U]$ (Lemma 2 (1)). Hence, by Lemma 4,

$$(u'v^2)^4 = (u'v^2)\{(u'v^2)^3 + v'(u'v^2)^2 + v'^2(u'v^2)\} = u'v^2(v + uv^2)^3 = 0$$

or

$$(u'v^2)^4 = (u'v^2)\{(u'v^2)^3 - v'(u'v^2)^2 + v'^2(u'v^2)\} = u'v^2(v - uv^2)^3 = 0.$$

Therefore $(u'v^2)^4 = 0$ for all $u \in U$. Now, choose $r \in R$ such that $r' \neq 0$. Then, $r'u = (ru)' - ru' \in U'$ for all $u \in U$, i.e., $r'U \subseteq U'$, and hence U' contains a nonzero ideal $Rr'U$. Since $Rr'Uv^2$ is a nil left ideal of bounded index, we get $v^2 = 0$ by Lemma 1 (5). But, this is impossible by Lemma 4 (2).

We are now ready to complete the proof of our theorem.

Proof of Theorem 1. For the convenience of notation, let us write $x^T = x'$.

(1) We put $W = U \cap U'$. Obviously, U' is a prime ring and W' is a nonzero ideal of U' . It is well known that $C' \subseteq C$. According to Corollary 1, T is centralizing on U . Now, by Jacobi's identity, we have $[[u, u''], u'] = 0$ for all $u \in U$, and so

$$\begin{aligned} [u, u'] [u'', u'] &= [u' [u, u''] + [u, u'] u'' + [u, u'] u' u'] \\ &= [[u, u' u''] + [u u', u'], u'] \\ &= [[u + u u', (u + u u')'], u'] = 0. \end{aligned}$$

Hence, $[x, x'] = 0$ for all $x \in U'$. Linearizing $[x, x'] = 0$ gives $[x, y'] = [x', y]$ for all $x, y \in U'$, and then

$$(x - x')[x, y'] = x[x, y'] - [x, x'y'] = x[x', y] - [x', xy] = 0.$$

Hence, noting that $z'[x, y'] = [x, (zy)'] - [x, z']y'$, we see that $(x - x')W[x, y'] = 0$ ($x, y \in W$). Then, since U' is a prime ring, we have $W = V_w(W') \cup K$, where $V_w(W')$ is the centralizer of W' in W and $K = \{x \in W \mid x' = x\}$. In view of Lemma 1 (2), $W \neq K$, and so $W = V_w(W')$ (by Brauer's trick). In particular, the nonzero ideal $W \cap W'$ of U' is commutative. Now, U' is commutative by Lemma 1 (1), and hence so is R by the same lemma.

(2) In view of Corollary 2 (1), T is centralizing on U . We consider the ring $R_1 = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in R \right\}$ with center $C_1 = \left\{ \begin{pmatrix} x & y \\ 0 & x \end{pmatrix} \mid x, y \in C \right\}$, where R is regarded as a subring of R_1 in an obvious way. As is easily seen, T gives rise to a ring homomorphism $x \rightarrow x^* = \begin{pmatrix} x & x' \\ 0 & x \end{pmatrix}$ of R into R_1 and $[u^*, u]$

$\in C_1$ for all $u \in U$. First, we claim that $[u', u] = 0$, or equivalently $[u^*, u] = 0$, for all $u \in U$. If R is of characteristic 2, then

$$\begin{aligned} 0 &= [[u + uu', (u + uu')'], u] = [[uu', u'] + [u, (uu')'], u] \\ &= [u', u]^2 + [u[u, u''], u]. \end{aligned}$$

Since $[u, u''] = [u, u']' \in C$, the last shows that $[u', u]^2 = 0$, and hence $[u', u] = 0$. On the other hand, if R is of characteristic not 2, then

$$4 \binom{0}{0} u^2 [u', u] = 2(u^*, u)[u^*, u] = [(u^2)^*, u^2] \in C_1,$$

i.e., $u^2[u', u] \in C$. Hence, $0 = [u', u^2[u', u]] = 2[u', u]^2 u$, and therefore $[u', u] = 0$.

Now, linearizing $[u^*, u] = 0$ gives $[u, v^*] = [u^*, v]$ for all $u, v \in U$, and then

$$(u - u^*)[u, v^*] = u[u, v^*] - [u, u^*v^*] = u[u^*, v] - [u^*, uv] = 0.$$

Hence, noting that $x^*[u, v^*] = [u, (xv)^*] - [u, x^*]v^*$ ($u, v \in U, x \in R$), we get $(u^* - u)x^*[u, v^*] = 0$, which becomes $u'x[u, v] = 0$, i.e., $u'R[u, v] = 0$. Thus, we get $U = V_U(U) \cup K$, where $K = \{u \in U \mid u' = 0\}$. Since $U \neq K$ by Lemma 1 (3), U coincides with its center, and therefore R is commutative by Lemma 1 (1).

Corollary 3. *Let U be a nonzero ideal of a prime ring R .*

(1) *Let T be a nontrivial ring endomorphism of R . If T induces a semicentralizing endomorphism of U , U^T is an ideal of U and $U^{T^2} \neq 0$, then R is commutative.*

(2) *Let T be a nontrivial derivation of R . If T induces a centralizing (resp. skew-centralizing) derivation of U , then R is commutative.*

Proof. U is a prime ring and T is nontrivial on U (Lemma 1 (2) and (3)). Hence, U is commutative by Theorem 1, and therefore so is R by Lemma 1 (1).

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(Received January 17, 1983)

Added in proof. After the submission of this paper, the authors received from Prof. J.H. Mayne an erratum sheet that corrects the proofs of [5, Theorem and Corollary] and a copy of his paper entitled “Centralizing mappings of prime rings” (submitted to *Canad. Math. Bull.*), where he has improved [5, Theorem].