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NOTE ON THE MAXIMAL RIGHT $\aleph_0$ -QUOTIENT RING OF A REGULAR RING

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Throughout, R will represent a ring with unity, and $\mathfrak{A}(R)$ (resp. $\mathfrak{B}(R)$) the set of all countably generated right ideals (resp. all countably generated, essential right ideals) of R . If a right R -module M is essential in N , we write $M \subseteq_e N$.

The present objective is to prove that the maximal right $\aleph_0$ -quotient ring of a regular ring R is a right $\aleph_0$ -continuous, regular ring if and only if R satisfies the $\aleph_0$ -complement property, i.e., every countably generated right ideal has a countably generated complement (Theorem 4). We also show that the $\aleph_0$ -complement property is Morita invariant. As for terminologies and fundamental results used in this note, we refer to Goodearl [1].

We begin with the following

Lemma 1 ([1, Prop. 14.11]). *Let R be a regular ring, and Q the maximal right quotient ring of R . Let $S = \{x \in Q : xJ \subset R \text{ for some } J \in \mathfrak{B}(R)\}$. Then there holds the following:*

- (1) *S is a subring of Q , and is the maximal right $\aleph_0$ -quotient ring of R .*
- (2) *For any $K \in \mathfrak{B}(R)$, every map in $\text{Hom}_S(K, S_S)$ extends to S .*

We say that R satisfies the *right $\aleph_0$ -complement property* if for any $I \in \mathfrak{A}(R)$ there exists some $J \in \mathfrak{A}(R)$ such that $I \cap J = 0$ and $I \oplus J \in \mathfrak{B}(R)$. Obviously, every regular ring having no uncountable direct sums of non-zero right ideals satisfies the $\aleph_0$ -complement property, and also every right $\aleph_0$ -continuous regular ring satisfies the same ([1, Lemma 14.8]). We shall give an example of a regular ring satisfying the $\aleph_0$ -complement property which is not right $\aleph_0$ -continuous and has an uncountable direct sum of non-zero right ideals.

Example (cf. [1, Example 13.8]). Let F be a field having a proper subfield K . Let I be an uncountable well-ordered set, and set $F_\alpha = F$, $K_\alpha = K$ ($\alpha \in I$). Let $Q = \prod_{\alpha \in I} F_\alpha$ and $R = \{x \in Q : x_\alpha \in K_\alpha \text{ for all but finitely many } \alpha\}$. Observing that R contains all the idempotents of Q , we see that R is

a continuous regular ring. Now, by Proposition 5 below, $T = M_2(R)$ satisfies the $\aleph_0$ -complement property. We identify T with the subring $\{A \in \prod_{\alpha \in I} M_2(F_\alpha) : A_\alpha \in M_2(K_\alpha) \text{ for all but finitely many } \alpha\}$. Evidently, T has an uncountable direct sum of non-zero right ideals. Since the lattice of principal right ideals of T is not $\aleph_0$ -complete by [1, Example 13.8], T is not right $\aleph_0$ -continuous.

Lemma 2. *Let R be a regular ring satisfying the right $\aleph_0$ -complement property, and S the maximal right $\aleph_0$ -quotient ring of R . Then, for any $I \in \mathfrak{A}(R)$ and any R -homomorphism $f: I \rightarrow R$, there exists an element s in S such that $f(a) = sa$ for all $a \in I$.*

Proof. Choose $J \in \mathfrak{A}(R)$ with $I \oplus J \subseteq_e R$, and extend f to a homomorphism $I \oplus J \rightarrow R$ by setting $f(J) = 0$. As is easily seen, $IS \oplus JS \subseteq_e S_S$. Then, noting that $K \otimes_R S \cong KS$ for any right ideal K of R ([1, Cor. 1.13]), we can define an S -homomorphism $f^*: IS \oplus JS \rightarrow S$ by setting $f^*(\sum_i a_i x_i + \sum_j b_j y_j) = \sum_i f(a_i) x_i$ ($a_i \in I$, $b_j \in J$, $x_i, y_j \in S$). Since $IS \oplus JS \subseteq_e S_S$, [1, Th. 14.11] shows that there exists $s \in S$ such that $f^*(x) = sx$ for all $x \in IS \oplus JS$, completing the proof.

Lemma 3. *If a regular ring R satisfies the right $\aleph_0$ -complement property, then so does the maximal right $\aleph_0$ -quotient ring S of R , and conversely.*

Proof. Let $K = \sum_n x_n S \in \mathfrak{A}(S)$. By definition, there exists $J_n \in \mathfrak{B}(R)$ such that $x_n J_n \subseteq R$ ($n = 1, 2, \dots$). Then, $I = \sum_n x_n J_n \in \mathfrak{A}(R)$. Consider the natural epimorphism $\bigoplus_n S/J_n \rightarrow K/I$. Inasmuch as $J_n \subseteq_e R \subseteq_e S_R$ for all n , we see that $(K/I)_R$ is singular, and therefore $I \subseteq_e K_R$. Then we have $I \subseteq_e K_S$. Choose $J \in \mathfrak{A}(R)$ such that $I \oplus J \subseteq_e R_R$. Then we have $IS \oplus JS \subseteq_e S_S$, and hence $K \oplus JS \subseteq_e S_S$.

Conversely, suppose that S has the right $\aleph_0$ -complement property. Given $C \in \mathfrak{A}(R)$, there exists $K = \sum_n x_n S \in \mathfrak{A}(S)$ such that $CS \oplus K \in \mathfrak{B}(S)$. By definition, we then have $CS \oplus K \subseteq_e S_R$. By making use of the above argument, we can find $I \in \mathfrak{A}(R)$ such that $I \subseteq_e K_R$. Now, combining this with $C \subseteq_e CS_R$, we get $C \oplus I \subseteq_e S_R$, and therefore $C \oplus I \in \mathfrak{B}(R)$.

We are now in a position to state our main theorem.

Theorem 4. *Let R be a regular ring, and S the maximal right $\aleph_0$ -quo-*

tient ring of R . Then the following conditions are equivalent :

- 1) R satisfies the right $\aleph_0$ -complement property.
- 2) S is a right $\aleph_0$ -continuous regular ring.

Proof. 1) $\Rightarrow$ 2). In view of Lemma 2, we can prove that S is regular, in the same way as in the proof of [1, Th. 14.12]. Now, let $K \in \mathfrak{U}(S)$. Then $K \oplus J \subseteq_e S_S$ with some $J \in \mathfrak{U}(S)$. According to Lemma 1 (2), the natural projection map $K \oplus J \rightarrow K$ is given by the left multiplication of an element p in S . Obviously, p is an idempotent and $K \subset pS$. Since $(pS/K)_S$ is singular as the homomorphic image of the singular module $(S/(K \oplus J))_S$, we see that $K \subseteq_e pS_S$, and hence S is right $\aleph_0$ -continuous by [1, Cor. 14.4].

2) $\Rightarrow$ 1). Obviously, S satisfies the right $\aleph_0$ -complement property. Hence, by Lemma 3, R satisfies the same property.

In general, the $\aleph_0$ -continuity of a regular ring is not Morita invariant. However, we shall show that the $\aleph_0$ -complement property of a regular ring is Morita invariant. We can prove the next in the same way as in the proof of [1, Lemma 14.18].

Lemma 5. *Let R be a regular ring with $\aleph_0$ -complement property, and P a finitely generated projective right R -module. If A is a countably generated submodule of P , then $A \oplus B \subseteq_e P$ with some countably generated submodule B of P .*

Proposition 6. *Let P be a finitely generated projective right module over a regular ring R . If R satisfies the $\aleph_0$ -complement property, then $\text{End}_R(P)$ does the same property.*

Proof. Put $T = \text{End}_R(P)$, and $M \in \mathfrak{U}(T)$. Then, by [1, Prop. 2.14], there exist orthogonal idempotents $f_1, f_2, \dots$ in T such that $M = \bigoplus_n f_n T$. We put $A = \bigoplus_n f_n(P)$, and choose a countably generated submodule B of P such that $A \oplus B \subseteq_e P$ (Lemma 5). In view of [1, Prop. 2.13], there exist countable idempotents $g_1, g_2, \dots$ in T such that $g_{n+1}g_n = g_n$ ($n=1, 2, \dots$) and $B = \bigcup_n g_n(P)$. Put $N = \bigcup_n g_n T \in \mathfrak{U}(T)$. It is easy to see that $M \cap N = 0$. Now, we shall show that $M \oplus N \subseteq_e T_T$. Let e be an arbitrary non-zero idempotent of T . Since $A \oplus B \subseteq_e P$ implies $(A \oplus B) \cap e(P) \neq 0$, we can find $x \neq 0$ in $(A \oplus B) \cap e(P)$. If α is a projection of P on the direct summand xR (see [1, Th. 1.11]), we can easily see that $\alpha \in (M + N) \cap eT$. We have thus completed the proof.

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