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On Higher Syzygies of Projective Toric Varieties

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Abstract

Let A be an ample line bundle on a projective toric variety X of dimension $n (\geq 2)$. It is known that the d-th tensor power A \otimes d embedds X as a projectively normal variety in Pr := P(H0(X,L \otimes d)) if $d \geq n \& #x2212$; 1. In this paper first we show that when dimX = 2 the line bundle A \otimes d satisfies the property Np for $p \leq 3d \& #x2212$; 3. Second we show that when dimX = $n \geq 3$ the bundle A \otimes d satisfies the property Np for $p \leq d \& #x2212$; n + 2 and $d \geq n \& #x2212$; 1.

KEYWORDS: toric variety, syzygy

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ON HIGHER SYZYGIES OF PROJECTIVE TORIC VARIETIES

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ABSTRACT. Let A be an ample line bundle on a projective toric variety X of dimension $n (\geq 2)$. It is known that the d-th tensor power $A^{\otimes d}$ embedds X as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X, L^{\otimes d}))$ if $d \geq n-1$. In this paper first we show that when dim X = 2 the line bundle $A^{\otimes d}$ satisfies the property N_p for $p \leq 3d-3$. Second we show that when dim $X = n \geq 3$ the bundle $A^{\otimes d}$ satisfies the property N_p for $p \leq d-n+2$ and $d \geq n-1$.

INTRODUCTION

The purpose of this article is to study the minimal free resolution of homogeneous coordinate rings of toric varieties.

Let X be a projective toric variety of dimension n and L a very ample line bundle on X. Since projective toric variety of dimension one is isomorphic to the projective line, we may assume that $n \ge 2$.

Koelman showed that an ample line bundle on a projective toric surface X is very ample and embedds X as a projectively normal variety in $\mathbb{P}^r := \mathbb{P}(H^0(X,L))$ [10], and obtained a criterion when the surface is defined by only quadrics [11]. When $n \geq 3$ an ample line bundle is not very ample in general. Ewald and Wessels [3] showed that for an ample line bundle A on X the d-th tensor power $L = A^{\otimes d}$ is very ample for $d \geq \dim X - 1$. Ogata and Nakagawa [13] showed that $L = A^{\otimes d}$ embedds X as a projectively normal variety if $d \geq \dim X - 1$ and that the homogeneous ideal I of X in $\mathbb{P}^r := \mathbb{P}(H^0(X,L))$ is generated by quadrics if $d \geq \dim X$. In this paper, we study higher syzygies of the homogeneous ideal of X in \mathbb{P}^r , especially the property N_p introduced by Green and Lazarsfeld [7].

Definition 1. Let X be a projective variety and L a very ample line bundle on X defining an embedding $X \hookrightarrow \mathbb{P}^r := \mathbb{P}(H^0(X,L))$. Denote by $S = \text{Sym } H^0(X,L)$ the homogeneous coordinate ring of the projective space \mathbb{P}^r . Consider the graded S-module $R = R(L) = \bigoplus_{i \ge 0} H^0(X, L^{\otimes i})$, the homogeneous coordinate ring of X. Let E_{\bullet} be a minimal graded free resolution of R:

 $\cdots \to E_2 \to E_1 \to E_0 \to R \to 0,$

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where $E_i = \bigoplus_j S(-a_{ij})$. Then the line bundle *L* satisfies *Property* (N_0) if $E_0 = S$. For an integer $p \ge 1$ the line bundle *L* satisfies *Property* (N_p) if $E_0 = S$ and if $a_{ij} = i + 1$ for $1 \le i \le p$.

Schenck and Smith [17] proved that for an ample line bundle A on a projective toric variety of dimension n, the bundle $A^{\otimes d}$ satisfies Property N_{d-n+1} for $d \ge n-1$.

This paper improves their results by separately considering the case n = 2 and $n \ge 3$.

Theorem 0.1. Let X be a projective toric surface and A an ample line bundle on X. Then $A^{\otimes d}$ satisfies Property N_p for $p \leq 3d - 3$.

This is given by Proposition 2.3.

Theorem 0.2. Let X be a projective toric variety of dimension $n (\geq 3)$ and A an ample line bundle on X. Then $A^{\otimes d}$ satisfies Property N_p for $p \leq d - n + 2$ and $d \geq n - 1$.

This is given by Proposition 3.3.

1. POLARIZED TORIC VARIETIES

First we mention the facts about toric varieties needed in this paper following Oda's book [14], or Fulton's book [5].

Let N be a free \mathbb{Z} -module of rank n, M its dual and $\langle , \rangle : M \times N \to \mathbb{Z}$ the canonical pairing. By scalar extension to the field \mathbb{R} of real numbers, we have real vector spaces $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$ and $M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$. Let $T_N := N \otimes_{\mathbb{Z}} \mathbb{C}^* \cong$ $(\mathbb{C}^*)^n$ be the algebraic *n*-torus over the field \mathbb{C} of complex numbers, where \mathbb{C}^* is the multiplicative group of \mathbb{C} . Then $M = \operatorname{Hom}_{\operatorname{gr}}(T_N, \mathbb{C}^*)$ is the character group of T_N . For $m \in M$ we denote $\mathbf{e}(m)$ the corresponding character of T_N . Let Δ be a complete finite fan of N consisting of strongly convex rational polyhedral cones σ , that is, there exist a finite number of elements v_1, v_2, \ldots, v_s in N such that

$$\sigma = \mathbb{R}_{\geq 0} v_1 + \dots + \mathbb{R}_{\geq 0} v_s,$$

and $\sigma \cap \{-\sigma\} = \{0\}$. Then we have a complete toric variety $X = T_N \operatorname{emb}(\Delta)$: = $\bigcup_{\sigma \in \Delta} U_{\sigma}$ of dimension n (see Section 1.2 [14], or Section 1.4 [5]). Here $U_{\sigma} = \operatorname{Spec} \mathbb{C}[\sigma^{\vee} \cap M]$ and σ^{\vee} is the dual cone of σ with respect to the pairing \langle, \rangle . For the origin $\{0\}$, the affine open set $U_{\{0\}} = \operatorname{Spec} \mathbb{C}[M]$ is the unique dense T_N -orbit. We note that a toric variety is always normal.

Let L be an ample T_N -equivariant invertible sheaf on X. Then the polarized variety (X, L) corresponds to an integral convex polytope P in $M_{\mathbb{R}}$ of dimension n. We call the convex hull $\text{Conv}\{u_0, u_1, \ldots, u_r\}$ in $M_{\mathbb{R}}$ of a

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finite subset $\{u_0, u_1, \ldots, u_r\} \subset M$ an *integral convex polytope* in $M_{\mathbb{R}}$. The correspondence is given by the isomorphism

(1.1)
$$H^0(X,L) \cong \bigoplus_{m \in P \cap M} \mathbb{C}\mathbf{e}(m),$$

where $\mathbf{e}(m)$ are considered as rational functions on X because they are functions on the open dense subset T_N of X (see Section 2.2 [14], or Section 3.5 [5]).

Let P_1 and P_2 be integral convex polytopes in $M_{\mathbb{R}}$. Then we can consider the Minkowski sum $P_1 + P_2 := \{x_1 + x_2 \in M_{\mathbb{R}}; x_i \in P_i \ (i = 1, 2)\}$ and the multiplication by scalars $rP_1 := \{rx \in M_{\mathbb{R}}; x \in P_1\}$ for a positive real number r. If l is a natural number, then lP_1 coincides with the l times sum of P_1 , i.e., $lP_1 = \{x_1 + \cdots + x_l \in M_{\mathbb{R}}; x_1, \ldots, x_l \in P_1\}$. The l-th tensor power $L^{\otimes l}$ corresponds to the convex polytope $lP := \{lx \in M_{\mathbb{R}}; x \in P\}$. Moreover the multiplication map

(1.2)
$$H^0(X, L^{\otimes l}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (l+1)})$$

transforms $\mathbf{e}(u_1) \otimes \mathbf{e}(u_2)$ for $u_1 \in lP \cap M$ and $u_2 \in P \cap M$ to $\mathbf{e}(u_1 + u_2)$ through the isomorphism (1.1). Therefore the equality $(lP \cap M) + (P \cap M) = (l+1)P \cap M$ is equivalent to the surjectivity of the multiplication map (1.2).

Proposition 1.1 (Nakagawa-Ogata [13]). Let X be a projective toric variety of dimension n and L an ample line bundle on X. Then the multiplication map

$$H^0(X, L^{\otimes i}) \otimes H^0(X, L) \to H^0(X, L^{\otimes (i+1)})$$

is surjective for all $i \ge n-1$.

This implies that $L^{\otimes d}$ satisfies Property N_0 for $d \ge n-1$. By employing an analogous method of Mumford [12] we obtained that $L^{\otimes d}$ satisfies Property N_1 for $d \ge n$ (see Corollary 2.2 in [13]). Schenck and Smith [17] generalizes for $L^{\otimes d}$ to satisfy Property N_p for $d \ge n-1+p$.

2. Toric Surfaces

Ogata[16] generalize the result of [11] to higher dimension by using the method of Fujita's regular ladder [4]. In this section we use the same method in the case of dimension two.

Let X be a projective toric surface and L an ample line bundle on X. We consider a general hyperplane section C of the linear system |L|. Since X is normal, we may assume that C is nonsingular. Set $L_C = L|C$, the restriction of L to the curve C. From easy calculation, we see that

(2.1)
$$h^0(C, L_C) = h^0(X, L) - 1 = {}^{\sharp}P \cap M - 1,$$

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(2.2)
$$h^1(C, \mathcal{O}_C) = h^2(X, L^{-1}) = h^0(X, \omega_X \otimes L) = {}^{\sharp} \operatorname{Int} P \cap M,$$

$$(2.3) \quad h^1(C, L_C) = 0$$

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Denote by $g = h^1(C, \mathcal{O}_C)$ the genus of C. Then from Riemann-Roch Theorem we have deg $L_C = 2g - 2 + {}^{\sharp} \partial P \cap M$.

Lemma 2.1 (Green [6], (4.a.1)). Let L be a line bundle of degree 2g+1+p $(p \ge 0)$ on a smooth irreducible projective curve of genus g. Then L satisfies Property N_p .

This is a generalization of Mumford [12] (the case p = 0) and of Fujita [4] (the case p = 1).

Lemma 2.2 (Green [6], (3.b.7)). Let X be a compact complex manifold, L a line bundle on X, $Y \subset X$ a connected hypersurface in the linear system |L| and L_Y denote the restriction of L to Y. Assume that

$$H^1(X, L^{\otimes q}) = 0 \quad for \ all \ q \ge 0.$$

Then $\mathcal{K}_{p,q}(X,L) \cong \mathcal{K}_{p,q}(Y,L_Y)$ for all p,q.

Since $H^1(X, L^{\otimes q}) = 0$ for $q \ge 0$ and for any ample ample line bundle Lon a toric variety X, this lemma implies that if L_Y satisfies Property N_p for a general smooth hypersurface Y in |L|, then L also satisfies Property N_p .

Proposition 2.3. Let A be an ample line bundle on a projective toric surface X corresponding to an integral convex polygon Q in $M_{\mathbb{R}}$ given by the isomorphism

$$H^0(X,A) \cong \bigoplus_{m \in Q \cap M} \mathbb{C}\mathbf{e}(m)$$

Then the d-th tensor power $A^{\otimes d}$ satisfies Property N_p for $p \leq d \ {}^{\sharp}Int \ Q \cap M-3$. In particular, $A^{\otimes d}$ satisfies Property N_p for $p \leq 3d-3$.

Proof. Set $L = A^{\otimes d}$. Let C be a general hyperplane section of |L|. Set $L_C = L|C$. Denote by g the genus of C. Then we have

$$\deg L_C = 2g - 2 + d \,^{\sharp} \operatorname{Int} \, Q \cap M$$
$$= 2g + 1 + (d \,^{\sharp} \operatorname{Int} \, Q \cap M - 3).$$

From Lemma 2.1, the bundle L_C satisfies Property N_p for $p \leq d \,^{\sharp}$ Int $Q \cap M - 3$. From Lemma 2.2 we obtain a proof of Proposition.

This is a generalization of the case $(X, A) = (\mathbb{P}^2, \mathcal{O}(1))$ treated by Birkenhake in [1].

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3. HIGHER DIMENSION

Lemma 3.1 (Ogata [15]). Let A be an ample line bundle on a projective toric variety of dimension $n \ (n \ge 3)$. Then $A^{\otimes d}$ satisfies Property N_1 for $d \ge n-1$.

A very ample invertible sheaf L on a projective variety X defines an embedding $\Phi_L : X \to \mathbb{P}(H^0(X, L)) = \mathbb{P}^r$. Set $M_L := \Phi_L^* \Omega_{\mathbb{P}^r}^1(1)$ so that there exists the following exact sequence of vector bundles

(3.1)
$$0 \to M_L \to H^0(X, L) \otimes_{\mathbb{C}} \mathcal{O}_X \to L \to 0.$$

Lemma 3.2 (Ein-Lazarsfeld [2]). Assume that L is very ample and that $H^1(X, L^{\otimes k}) = 0$ for all $k \geq 1$. Then L satisfies Property N_p if and only if

$$H^1(X, \wedge^a M_L \otimes L^{\otimes b}) = 0 \quad for \ 1 \le a \le p+1 \ and \ b \ge 1.$$

Since in characteristic zero $\wedge^a M_L$ is a direct summand of $M_L^{\otimes a}$, we have only to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ as in [2] and [8].

Proposition 3.3. Let $p \ge 2$ be an integer. Let A be an ample line bundle on a projective toric variety X of dimension $n \ (n \ge 3)$. Then $A^{\otimes d}$ satisfies Property N_p for $p \le n-2-d$.

For a proof we have to show the vanishing of $H^1(X, M_L^{\otimes a} \otimes L^{\otimes b})$ for $1 \leq a \leq p+1$ and $b \geq 1$ with $L = A^{\otimes d}$. We need a lemma.

Lemma 3.4 (Mumford [12]). Let F be a coherent sheaf on a projective algebraic variety X. Let A be a line bundle on X generated by global sections. If $H^i(X, F \otimes A^{\otimes (-i)}) = 0$ for all $i \ge 1$, then the multiplication map

$$H^0(X, F \otimes A^{\otimes j}) \otimes H^0(X, A) \to H^0(X, F \otimes A^{\otimes (j+1)})$$

is surjective for all $j \ge 0$.

For a proof see Theorem 2 in [12].

Proof of Proposition 3.3. Let $q \ge 2$ be an integer and $L = A^{\otimes d}$ with $d \ge n + q - 2$. We want to show that

(3.2) $H^1(X, M_L^{\otimes q} \otimes A^{\otimes j}) = 0 \text{ for } j \ge n+q-3,$

(3.3) $H^{i}(X, M_{L}^{\otimes q} \otimes A^{\otimes j}) = 0 \text{ for } i \geq 2 \text{ and } j \geq 0.$

From Proposition 1.1 and the exact sequence (3.1) we see that

(3.4)
$$H^{i}(X, M_{L} \otimes A^{\otimes j}) = 0 \quad \text{for } i \ge 1, j \ge 0 \text{ and } d \ge n-1.$$

First we shall show the vanishing of (3.2) for q = 2. Taking tensor product of (3.1) with $M_L \otimes A^{\otimes j}$ we obtain an exact sequence

$$(3.5) \quad 0 \to M_L^{\otimes 2} \otimes A^{\otimes j} \to H^0(X,L) \otimes_{\mathbb{C}} M_L \otimes A^{\otimes j} \to M_L \otimes L \otimes A^{\otimes j} \to 0.$$

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From (3.4) and (3.5) we have

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(3.6)
$$H^i(X, M_L^{\otimes 2} \otimes A^{\otimes j}) = 0 \quad \text{for } i \ge 2, \ j \ge 0 \text{ and } d \ge n-1.$$

Second we shall show the vanishing of (3.2) for q = 2. From Lemmas 3.1 and 3.2, we see that $H^1(X, \wedge^2 M_L \otimes L^{\otimes b}) = 0$ for $b \ge 1$ and $d \ge n-1$. Taking wedge product in (3.1) and twisting by $L^{\otimes b}$, we obtain an exact sequence

$$(3.7) \qquad 0 \to \wedge^2 M_L \otimes L^{\otimes b} \to \wedge^2 H^0(X,L) \otimes_{\mathbb{C}} L^{\otimes b} \to M_L \otimes L^{\otimes (b+1)} \to 0.$$

The vanishing of the first cohomology group implies the surjectivity of the map

(3.8)
$$\wedge^2 H^0(X,L) \otimes H^0(X,L^{\otimes b}) \to H^0(X,M_L \otimes L^{\otimes (b+1)}).$$

Taking global sections of (3.7) with b = 0 we see that the surjective map (3.8) factors through $\wedge^2 H^0(X, L) \to H^0(X, M_L \otimes L)$. Thus we have that

(3.9)
$$H^1(X, M_L^{\otimes 2} \otimes L^{\otimes b}) = 0 \text{ for } b \ge 1 \text{ and } d \ge n-1$$

from the exact sequence (3.5) replacing A by L. For line bundles L_1 and L_2 , denote by $R(L_1, L_2)$ the kernel of the multiplication map $H^0(X, L_1) \otimes H^0(X, L_2) \to H^0(X, L_1 \otimes L_2)$. By taking a global section of the exact sequence

$$0 \to M_L \otimes A \to \Gamma(X, L) \otimes_{\mathbb{C}} A \to L \otimes A \to 0,$$

we have $H^0(X, M_L \otimes A) = R(L, A)$. Then we can rewrite the sequence (3.5) after taking its global sections as

$$\begin{array}{cccc} H^0(X,L) \otimes H^0(X,M_L \otimes A^{\otimes j}) & \longrightarrow & H^0(X,M_L \otimes L \otimes A^{\otimes j}) \\ & \parallel & & \parallel \\ & & H^0(X,L) \otimes R(A^{\otimes j},L) & \longrightarrow & R(L \otimes A^{\otimes j},L). \end{array}$$

For simplicity we denote $A^{\otimes i}$ as A^i and $H^0(X, L)$ as $\Gamma(L)$. From Corollary 2.2 in [13], we have that

$$\Gamma(A^i) \otimes R(A^d, A^j) \to R(A^{d+i}, A^j)$$

is surjective for $d \ge n$, $i \ge 1$ and $j \ge 1$. Hence we showed the vanishing of (3.2) for $j \ge n$. We remain to show the vanishing of $H^1(X, M_L^{\otimes 2} \otimes A^{\otimes (n-1)})$

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for $d \ge n$. Consider the diagram

If α is surjective, then β is also surjective. The vanishing (3.9) implies the surjectivity of

$$R(A^{n-1}, A^{n-1}) \otimes \Gamma(A^{n-1}) \to R(A^{n-1}, A^{2n-2}).$$

Since $\Gamma(A^i) \otimes \Gamma(A^{n-1}) \to \Gamma(A^{n+i-1})$ is surjective for $i \ge 1$, the map α is surjective, hence,

$$\Gamma(A^d) \otimes R(A^{n-1}, A^d) \to R(A^{d+n-1}, A^d)$$

is surjective. This map is the same as

$$\Gamma(L) \otimes \Gamma(M_L \otimes A^{n-1}) \to \Gamma(M_L \otimes L \otimes A^{n-1})$$

with $L = A^d$. Thus we obtain that $H^1(X, M_L^{\otimes 2} \otimes A^{n-1}) = 0$. For $q \ge 2$ and $L = A^d$ with $d \ge n + q - 2$, if we have the equalities (3.2) and (3.3), then from the exact sequence

$$(3.10) \quad 0 \to M_L^{\otimes (q+1)} \otimes A^j \to \Gamma(L) \otimes M_L^{\otimes q} \otimes A^j \to M_L^{\otimes q} \otimes L \otimes A^j \to 0$$

we have

$$H^i(X, M_L^{\otimes (q+1)} \otimes A^j) = 0 \text{ for } i \ge 2 \text{ and } j \ge 0,$$

and from Lemma 3.4 we see the surjectivity of

$$\Gamma(M_L^{\otimes q} \otimes A^j) \otimes \Gamma(A) \to \Gamma(M_L^{\otimes q} \otimes A^{j+1})$$

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for $j \ge n + q - 2$. From this and the exact sequence (3.10) we have

$$H^1(X, M_L^{\otimes (q+1)} \otimes A^j) = 0 \quad \text{for } j \ge n+q-2.$$

In particular, we have

$$H^1(X, M_L^{\otimes (q+1)} \otimes L^{\otimes b}) = 0$$

for $b \ge 1$ and $L = A^{\otimes d}$ with $d \ge n + q - 2$.

References

- C. Birkenhake, *Linear systems on projective spaces*, manuscripta math. 88 (1995), 177–184.
- [2] L. Ein and R. Lazarsfeld, Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension, Inventiones Mathematicae 111 (1993), 51–67.
- [3] G. Ewald and U. Wessels, On the ampleness of invertible sheaves in complete projective toric varieties, Results in Mathematics 19 (1991), 275–278.
- [4] T. Fujita, Defining Equations for Certain Types of Polarized Varieties in Complex Analysis and Algebraic Geometry (edited by Bailly and Shioda) Iwanami and Cambridge Univ. Press, 1977, pp.165–173.
- [5] W. Fulton, Introduction to Toric Varieties, Ann. of Math. Studies No. 131, Princeton Univ. Press, 1993.
- [6] M. Green, Koszul cohomology and the geometry of projective varieties, J. Diff. Geometry 20 (1984), 125–171.
- [7] M. Green and R. Lazarsfeld, Some results on the syzygies of finite sets and algebraic curves, Compositio Math. 67 (1988), 301–314.
- [8] F. J. Gallego and B. P. Purnaprajna, *Higher syzygies of elliptic ruled surfaces*, J. Algebra 186 (1996), 626–659.
- [9] M. Hering, Syzygies, regularity and toric varieties, preprint AG/0402328(2004).
- [10] R. J. Koelman, Generators for the ideal of a projectively embedded toric surfaces, Tohoku Math. J. 45 (1993), 385–392.
- [11] _____, A criterion for the ideal of a projectively embedded toric surfaces to be generated by quadrics, Beiträger zur Algebra und Geometrie **34** (1993), 57–62.
- [12] D. Mumford, Varieties defined by quadric equations, In : Questions on Algebraic Varieties, Corso CIME (1969), 30–100.
- [13] K. Nakagawa and S. Ogata, On generators of ideals defining projective toric varieties, manuscripta math. 108 (2002), 33–42.
- [14] T. Oda, Convex Bodies and Algebraic Geometry, Ergebnisse der Math. 15, Springer-Verlag, Berlin, Heidelberg, New York, London, Paris, Tokyo, 1988.
- [15] S. Ogata, Quadratic generation of ideals defining projective toric varieties, Kodai Math. J. 26 (2003), 137–146.
- [16] _____, On projective toric varieties whose defining ideals have minimal generators of the highest degree, Ann. Institute Fourier **53** (2003), 2243–2255.
- [17] H. Schenck and G. G. Smith, Syzygies of projective toric varieties, preprint, AG/0308205(2003).

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