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ON WEAKLY PERIODIC RINGS, PERIODIC RINGS AND COMMUTATIVITY THEOREMS

JULIE GROSEN, HISAO TOMINAGA and ADIL YAQUB

Throughout, R will represent a ring with center Z . Let N, E be the set of nilpotent elements of R and the set of idempotents of R , respectively; let N^* be the subset of N consisting of all elements in R which square to zero. For each integer $n > 1$, we put $E_n = \{x \in R \mid x^n = x\}$. Call an element x of R *potent* if $x \in P = \bigcup_{n=2}^{\infty} E_n$. A ring R will be called *periodic* if for each $x \in R$, there exist distinct positive integers n, m for which $x^n = x^m$. By [4, Proposition 2], R is periodic if and only if for each $x \in R$, there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $x - f(x) \in N$. If every element of R is expressible as a sum of a potent element and a nilpotent element: $R = P + N$, R is called a *weakly periodic ring*. It is well-known that if R is periodic then it is weakly periodic (see, e.g. [1]). Whether R is weakly periodic implies that R is periodic is apparently not known, except in the presence of additional hypotheses ([2], [3] and [7]).

The major purpose of this paper is to prove the following theorems.

Theorem 1. *Let P be an h -property, that is, a ring-property which is inherited by every subring and every homomorphic image. Then the following statements are equivalent:*

- 1) *For any weakly periodic ring satisfying P , its commutator ideal is nil.*
- 2) *For every prime p , $M_2(\text{GF}(p))$ fails to satisfy P .*

Theorem 2. *Let R be a ring. Suppose that $R = \langle E \cup N \rangle$. If N is an ideal of R and $nE = 0$ for some positive integer n , then R is periodic.*

Theorem 3. *Let R be a weakly periodic ring with 1, and let D_r (resp. D_l) be the set of right (resp. left) zero-divisors of R . Suppose that N is commutative and $D_r \subseteq E + N$. Then N is an ideal of R and $\bar{R} = R/N$ is either Boolean or a field.*

Theorem 4. *Let R be a weakly periodic ring. Suppose that 1) for each $x \in R$ there exists $f(X) \in X^2\mathbb{Z}[X]$ such that $x - f(x) \in C_R(N)$, and 2) for each $x \in N + E_n$ ($n > 1$) and $a \in N$, $[(ax + x)^{n+1} - (xa + x)^{n+1}, x] = 0$.*

Then R is commutative.

Theorem 5. Let R be a ring with 1, and $n > 1$ a fixed integer. Suppose that $[x - x^n, y - y^n] = 0$ for all $x, y \in R$.

(1) Let Q be the intersection of the set of non-units of R with the set of quasi-regular elements of R . If $(xy)^n - x^n y^n \in Z$ for all $x, y \in R \setminus Q$, and $(n-1)[a, x] = 0$ implies $[a, x] = 0$ for all $a \in N$, $x \in R$, then R is commutative.

(2) If $(xy)^n - (yx)^n \in Z$ for all $x, y \in R$, then R is commutative.

(3) If $[x^n, y^n] \in Z$ for all $x, y \in R$, then R is commutative.

Obviously, Theorem 3 generalizes [5, Theorem 3.3], and Theorem 4 shows that [5, Theorems 2.1, 2.2 and 2.3] are still valid without the hypothesis that R is a periodic ring.

In advance of proving our theorems, we state three lemmas.

Lemma 1. Let R be a weakly periodic ring. Then the Jacobson radical J of R is nil. If, furthermore, $xR \subseteq N$ for all $x \in N$, then $N = J$ and R is periodic.

Proof. Let x be an arbitrary element of J , and write $x = b - a$, where $b^n = b$ ($n > 1$) and $a \in N$. Then $x + a = b = b^n = (x + a)^n$. Noting that x is in J , we see that $a - a^n \in J$, whence $a \in J$ follows. This proves that $b^n = b = x + a \in J$. Since b^{n-1} is an idempotent with $b = b^{n-1}b$, we get $b = 0$, so that $x = -a \in N$. The latter assertion is almost clear.

Lemma 2. If R is a weakly periodic division ring, then it is a field.

Proof. Obviously, for each $x \in R$ there exists an integer $n > 1$ such that $x^n = x$. Hence R is commutative, by Jacobson's theorem.

Lemma 3. Suppose that R satisfies the following condition:

(C) for each $x, y \in R$ there exist $f(X), g(X)$ in $X^2\mathbb{Z}[X]$ such that $[x - f(x), y - g(y)] = 0$.

If for each $a \in N^*$ and $x \in R$, there exists a positive integer k such that $[a, x]_k (= [[a, x]_{k-1}, x]) = 0$, then R is commutative.

Proof. By [6, Theorem C and Lemma 1 (2)], we see that $[a, x] = 0$ for all $a \in N^*$ and $x \in R$. Hence, [6, Lemma 2] shows that $R = C_R(N^*)$ is commutative.

Proof of Theorem 1. Since $M_2(\text{GF}(p))$ is a periodic ring, it remains only to prove that 2) implies 1). Let J be the Jacobson radical of a weakly periodic ring R satisfying P , and let R' be a primitive homomorphic image of R . If R' is not a division ring then, by the structure theorem of primitive rings, we can easily see that there exists a prime p such that $M_2(\text{GF}(p))$ is a factorsubring of R' . But this contradicts 2). Hence R' has to be a division ring, so R' is a field by Lemma 2. We have thus seen that R/J is commutative, that is the commutator ideal $C(R)$ of R is contained in J . Hence, by Lemma 1, $C(R)$ is nil, which proves the theorem.

Corollary 1. *Let P be an h -property. Suppose that for each prime p , $M_2(\text{GF}(p))$ fails to satisfy P . Then every weakly periodic ring satisfying P is periodic.*

Corollary 2. *Let R be a weakly periodic ring, and $m > 1$ an integer. Suppose that for each $x_1, \dots, x_m \in R$, there exists a monic monomial (word) w and a polynomial f in $\mathbb{Z}\langle X_1, \dots, X_m \rangle$ such that*

$$[x_1 w(x_1, \dots, x_m) x_m - x_m f(x_1, \dots, x_m) x_1, x_1] = 0.$$

Then R is periodic.

Proof of Theorem 2. As is well-known, R/N is a subdirect sum of subdirectly irreducible rings R_i ($i \in I$). Since R/N is generated by central idempotents as ring, R_i is generated by an identity element as ring. Noting that $nE = 0$, we see that R_i is a homomorphic image of $\mathbb{Z}/n\mathbb{Z}$. It is easy to see that $\mathbb{Z}/n\mathbb{Z}$ satisfies the polynomial identity $X^{2k} - X^k = 0$ for some positive integer k . Also R/N satisfies the same identity. Thus, R is periodic by [4, Proposition 2].

Corollary 3. *Suppose that $R = \langle E \cup N \rangle$ and $nE = 0$ for some positive integer n . If N is commutative then R is periodic. In particular, if R satisfies the condition (C) then R is periodic.*

Proof. By [2, Theorem 2], N is a commutative ideal. The latter assertion is clear by [6, Theorem C].

Proof of Theorem 3. In view of [2, Theorem 2], N is an ideal of R and R is periodic by [4, Proposition 2]. As is easily seen, $D_r = D_l = R \setminus U$, where U is the set of units in R . Hence $\bar{R} = \bar{E} \cup \bar{U}$ is commutative,

by Jacobson's theorem. Suppose now that there exists an idempotent $\bar{e} \neq 0, 1$ in $\bar{R}$. Then, for each $\bar{x} \in \bar{R}$, $\bar{e}\bar{x} \in \bar{U}$ and $(1-\bar{e})\bar{x} \in \bar{U}$. Hence $\bar{e}\bar{x}^2 = (\bar{e}\bar{x})^2 = \bar{e}\bar{x}$ and $(1-\bar{e})\bar{x}^2 = (1-\bar{e})\bar{x}$, whence $\bar{x}^2 = \bar{x}$ follows.

Proof of Theorem 4. By 1), we can easily see that N is commutative. Hence, by [2, Theorem 2], N is a commutative ideal.

Now, let $x \in N + E_n$ ($n > 1$) and $a \in N$. Then $x - x^n \in N$ and $x^2 - x^{n+1} \in N$. In particular, R satisfies the condition (C). Further, noting that $N^2 \subseteq Z$, we have

$$[[a, x^{n+1}], x] = [(ax+x)^{n+1} - (xa+x)^{n+1}, x] = 0.$$

Combining this with $[a, x^2 - x^{n+1}] = 0$, we get $[[a, x^2], x] = 0$. Hence, by [6, Lemma 1 (2)],

$$[a, x^2] = 0 \text{ for all } x \in R \text{ and } a \in N.$$

In particular, this proves that $E \subseteq Z$.

The usual argument then shows that we may assume, without loss of generality, that R is a local ring with radical N and characteristic p^a for some prime p (see [1, Lemma 1 (d)]). In order to see that R is commutative, it is enough to show that $[a, x] = 0$ for all $x \in R$ and $a \in N$ (see Lemma 3). Obviously, $2[a, x] = [a, (x+1)^2] = 0$. In case $p \neq 2$, $[a, x] = 0$ is immediate. Henceforth, we assume $p = 2$. Further, we may assume that $x \notin N$. Then, by $x^n - x \in N$ with some $n > 1$, $\bar{x} = x + N$ generates a finite field $\text{GF}(2^k)$. Hence $[a, x^{2^k} - x] = 0$, and therefore $[a, x] = [a, x^{2^k}] = [a, (x^2)^{2^{k-1}}] = 0$.

Proof of Theorem 5. In view of Lemma 3, it is enough to show that $[a, x]_3 = 0$ for all $a \in N^*$ and $x \in R$. By [6, Theorem C], N is a commutative ideal, so $N^2 \subseteq Z$. If $a \in N$ and $x \in R$, then $[a, x - x^n] = [a - a^n, x - x^n] = 0$, and hence $[a, x] = [a, x^n]$.

(1) Let $a \in N^*$. If $x \in R \setminus Q$ then, since $N^2 \subseteq Z$, $(n-1)[a, x]_2 = [(n-1)[a, x^n], x] = [(x(1+a))^n - x^n(1+a)^n, x] - [((1+a)x)^n - (1+a)^n x^n, x] = 0$, and therefore $[a, x]_2 = 0$. If $x \in Q$, then $1-x \in R \setminus Q$, so $[a, x]_2 = [a, 1-x]_2 = 0$, by the above.

(2) Let $a \in N^*$, $x \in R$. Then, since $N^2 \subseteq Z$, $[a, x]_2 = [[a, x^n], x] = [((1+a)x)^n - (x(1+a))^n, x] = 0$.

(3) Let $a \in N^*$, $x \in R$. Then, since $N^2 \subseteq Z$, $[a, x]_3 = [[a, x], x^n]_2 = [[a, x^n], x^n]_2 = [(x+ax)^n, x^n] - [(x+xa)^n, x^n], x^n] = 0$.

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