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## A Note on Anderson-Anderson-Johnson Questions

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## A NOTE ON ANDERSON-ANDERSON- JOHNSON QUESTIONS

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In this paper a ring means a commutative integral domain. Also dimension means Krull dimension. A ring  $A$  is called a locally factorial ring if  $A_M$  is a factorial ring (that is, a unique factorization domain) for each maximal ideal  $M$  of  $A$ . Let  $X^1(A)$  be the set of height one prime ideals of  $A$ .

Let  $A$  be a Krull ring. [2, § 2] asks the following questions :

(Q-1) Let  $P$  be a non-zero prime ideal of  $A$ . If  $P \cap Q = PQ$  for each prime ideal  $Q$  of  $A$  with  $\text{ht}(Q) \leq 2$  that is incomparable with  $P$ , then is  $P$  a maximal ideal or an invertible ideal of  $A$  ?

(Q-2) Are the following conditions equivalent ?

- (1)  $A$  is a locally factorial ring with  $\dim(A) \leq 2$ .
- (2)  $P \cap Q = PQ$  for each incomparable prime ideals  $P, Q$  of  $A$ .
- (3) Each ideal of  $A$  is a product of primary ideals of  $A$ .

Let  $B$  be a Noetherian ring with quotient field  $L$ ,  $K$  a finite algebraic extension field of  $L$  and  $A$  the integral closure of  $B$  in  $K$ . By Mori-Nagata's Integral Closure Theorem,  $A$  is a Krull ring. Such rings are important examples of Krull rings. We will call these rings Krull rings of Mori-Nagata type. The aim of this note is to answer (Q-1) and (Q-2) for Krull rings of Mori-Nagata type. We will prove that the answers for (Q-1) and (Q-2) are 'yes' for Krull rings of Mori-Nagata type.

**Lemma 1** (Mori-Nagata's Integral Closure Theorem [5, (33. 10) Theorem]). *Let  $B$  be a Noetherian ring,  $L$  the quotient field of  $B$ ,  $K$  a finite algebraic extension of  $L$  and  $A$  the integral closure of  $B$  in  $K$ . Then*

- (1)  $A$  is a Krull ring.
- (2) Let  $P$  be a prime ideal of  $A$  and  $P' = P \cap B$ . Then the quotient field of  $A/P$  is a finite algebraic extension of that of  $B/P'$ .
- (3) For each prime ideal  $P'$  of  $B$ , there exists only a finite number of prime ideals of  $A$  lying over  $P'$ .

**Lemma 2** (Nagata's Theorem [5, (33. 12) Theorem]). *Let  $A$  be a Krull ring of Mori-Nagata type. If  $\dim(A) \leq 2$ , then  $A$  is a Noetherian ring.*

**Lemma 3** (Ratliff [6, Lemma 2. 1]). *Let  $A$  be a Noetherian semilocal ring,  $P$  a prime ideal of  $A$ . Then there exists only a finite number of prime ideals  $Q$  of  $A$  such that  $Q \supset P$ ,  $\text{ht}(Q/P) = 1$  and  $\text{ht}(Q) \geq \text{ht}(P) + 2$ .*

The proofs of the following Lemma 4 and Lemma 5 appear on [4]. But we will prove them here again for convenience.

**Lemma 4** (Goto). *Let  $A$  be a Krull ring of Mori-Nagata type, and  $P$  a prime ideal of  $A$ . Then we have  $\bigcap_{i=1}^{\infty} P^i A_P = (0)$ .*

*Proof.* Let  $B$ ,  $L$  and  $K$  be as in Lemma 1. Set  $P \cap B = P'$  and  $B - P' = S$ . Then  $A_S$  is the integral closure of  $B_S$  in  $K$ . Thus we may assume that  $B$  is a Noetherian local ring and  $P \cap B$  is the maximal ideal of  $B$ . We may use the induction on  $\dim(B)$ . Set  $\dim(B) = d$ . If  $d = 0, 1$ , then the statement is clear. Suppose that  $d \geq 2$  and the statement holds for all positive integers lower than  $d$ . Set  $[K : L] = n$ . Then  $K = \sum_{i=1}^n Lx_i$  for some elements  $x_1, x_2, \dots, x_n$  of  $A$ . Hence we may assume that  $K = L$ . Also we may assume that  $\text{ht}(P) \geq 2$ . Set  $\{Q \in X^u(A) ; Q \subset P\} = \Sigma$ . Let  $Q \in \Sigma$  and set  $Q \cap B = Q'$ . Then  $Q' \neq (0)$ ,  $B/Q' \subset A/Q$  and the quotient field of  $A/Q$  is a finite algebraic extension of that of  $B/Q'$  by Lemma 1(2). Let  $R$  be the integral closure of  $B/Q'$  in the quotient field of  $A/Q$ . We have  $\dim(B/Q') \leq d-1$ . Choose a prime ideal  $N$  of  $R$  such that  $N \cap (A/Q) = P/Q$ . Since  $\dim(R) \leq d-1$ , we have

$$(0) = \bigcap_{i=1}^{\infty} N^i R_N \supset \bigcap_{i=1}^{\infty} P^i (A_P/QA_P).$$

Thus  $\bigcap_{i=1}^{\infty} P^i A_P \subset \bigcap_{i=1}^{\infty} QA_P$ . Since  $\dim(A_P) \geq 2$ , it follows that  $\bigcap_{Q \in \Sigma} QA_P = (0)$ . Thus  $\bigcap_{i=1}^{\infty} P^i A_P = (0)$ .

**Lemma 5** ([4, (4. 2)]). *Let  $A$  be a Krull ring of Mori-Nagata type, and  $P \in X^u(A)$ . If  $P \cap Q = PQ$  for each  $Q \in X^u(A)$  such that  $Q \neq P$ , then  $P$  is an invertible ideal of  $A$ .*

*Proof.* Let  $M$  be a maximal ideal of  $A$  containing  $P$ . Set  $A_M = A'$ ,  $MA_M = M'$  and  $PA_M = P'$ . It suffices to show that  $P'$  is a principal ideal of  $A'$  (cf. [1, Corollary 2. 2]). If  $P' = P'M'$ , then

$$P' = P'M' = P'M'^2 = P'M'^3 = \dots = (0)$$

by Lemma 4 ; a contradiction. Therefore  $P' \not\subseteq P'M'$ . Since  $A'$  is a Krull ring,  $A'$  is an atomic ring, that is, each element of  $A'$  is a product of irreducible elements of  $A'$ . Choose  $a' \in P' - P'M'$ . Then  $a'$  is an irreducible element of  $A'$ . Because, if  $a'$  is a reducible element with  $a' =$

$p'_1 p'_2 \cdots p'_n$ , where  $n \geq 2$  and each  $p'_i$  is irreducible element, then  $p'_i \in P'$  for some  $i$ . Thus  $a' \in P'M'$ ; a contradiction. Let  $Q' \in X^0(A')$  which is distinct with  $P'$ . If  $a' \in Q'$ , then  $a' \in P' \cap Q' = P'Q' \subset P'M'$ ; a contradiction. Thus  $P'$  is the only height one prime ideal containing  $a'$ . Let  $v$  be the valuation of the quotient field of  $A'$  with the valuation ring  $A'_v$ . Then we have  $v(a') = v(b')$  for each  $b' \in P' - P'M'$ . Because if, say,  $v(a') \leq v(b')$ , then there exists  $x'$  of  $M'$  such that  $x' = b'/a'$ . Then  $b' = a'x' \in P'M'$ ; a contradiction. Thus  $P' = a'A' \cup P'M'$ . It follows that  $P' = a'A'$ , namely a principal ideal of  $A'$ .

The idea of the proof of the following Theorem 6 is similar with that of [2, Theorem 2. 1].

**Theorem 6.** *Let  $A$  be a Krull ring of Mori-Nagata type. Let  $P$  be a non-zero prime ideal of  $A$ . If  $P \cap Q = PQ$  for each prime ideal  $Q$  of  $A$  with  $\text{ht}(Q) \leq 2$  that is incomparable with  $P$ , then  $P$  is a maximal ideal or an invertible ideal of  $A$ .*

*Proof.* Let  $B$ ,  $L$  and  $K$  be as in Lemma 1. Choose a maximal ideal  $M$  of  $A$  which contains  $P$ . Set  $B \cap M = M'$ . By Lemma 1(3) only a finite number of maximal ideals  $M_1, M_2, \dots, M_n$  of  $A$  lie over  $M'$ . We set  $M = M_1$ . Choose  $s_2 \in M_2 - M, \dots, s_n \in M_n - M$ . Set  $B[s_2, \dots, s_n] = B_0$ . Then  $M$  is the only maximal ideal of  $A$  lying over  $B_0 \cap M$ . Therefore we may assume that  $M$  is the only maximal ideal of  $A$  lying over  $M'$ . Set  $B - M' = S$ . Then  $A_S$  is the integral closure of  $B_S$  in  $K$ . We have  $P_S \cap Q = P_S Q$  for each prime ideal  $Q$  of  $A_S$  with  $\text{ht}(Q) \leq 2$  and incomparable with  $P_S$ . Moreover  $M_S$  is the only maximal ideal of  $A_S$  lying over  $M'_S$ . If  $P_S$  is an invertible ideal of  $A_S$ , then  $P$  is an invertible ideal of  $A$  by Lemma 5 and if  $P_S$  is a maximal ideal of  $A_S$ , then  $P$  is a maximal ideal of  $A$ . Therefore we may assume that  $A$  is a quasi-local ring with the maximal ideal  $M$ ,  $B$  is a Noetherian local ring with the maximal ideal  $M'$  and  $M \cap B = M'$ . We assume that  $P$  is neither maximal nor invertible. We will derive a contradiction. We have  $P \not\subseteq PM$  by Lemma 4. Choose  $x \in P - PM$ . Then  $B[x]$  is a Noetherian local ring. We set  $M \cap B[x] = M''$ ,  $P \cap B[x] = P''$  and  $P'' \cap B = P'$ . Choose a prime ideal  $Q''$  of  $B[x]$  which is minimal over  $xB[x]$  and contained in  $P''$ . We have  $\text{ht}(Q'') = 1$ . Choose a prime ideal  $Q$  of  $A$  lying over  $Q''$ . Then we have  $\text{ht}(Q) = 1$ . If  $P$  and  $Q$  are incomparable, then  $x \in P \cap Q = PQ \subset PM$ ; a contradiction. Therefore  $P$  and  $Q$  are comparable. Since  $\text{ht}(P) \geq 2$ , we have  $P \supset Q$ . By Lemma 3, there exists only a finite number

of prime ideals  $N''$  of  $B[x]$  such that  $\text{ht}(N''/Q'') = 1$  and  $\text{ht}(N'') \geq 3$ . Denote them by  $N''_1, \dots, N''_t$ . We have  $M'' \subset P'' \cup N''_1 \cup \dots \cup N''_t$ . Choose an element  $y$  of  $B[x]$  such that  $y \in M''$  and  $y \notin P'' \cup N''_1 \cup \dots \cup N''_t$ . Choose a prime ideal  $Q''_0/Q''$  of  $B[x]/Q''$  which is minimal over the element  $y+Q''$ . Then  $y \in Q''_0$ . We have  $\text{ht}(Q''_0/Q'') = 1$ . Since  $y \notin N''_i$ , we see that  $Q''_0 \neq N''_i$  ( $1 \leq i \leq t$ ). Therefore  $\text{ht}(Q''_0) = 2$ . Choose a prime ideal  $Q_0$  of  $A$  containing  $Q$  and lying over  $Q''_0$ . We have  $\text{ht}(Q_0) = 2$ . Since  $y \notin P''$ , also we have  $Q_0 \not\subset P$ . If  $P$  and  $Q_0$  are comparable, then  $\text{ht}(P) \leq \text{ht}(Q_0) - 1$  and hence  $\text{ht}(P) = 1$ ; a contradiction. Therefore  $P$  and  $Q_0$  are incomparable. Hence we have  $x \in P \cap Q_0 = PQ_0 \subset PM$ ; which is a contradiction.

**Remark 7** ([4, (3. 5)]). Let  $A$  be a Krull ring. Suppose that  $(Q-1)$  is 'yes' for  $A$ . Let  $P \in X^{(1)}(A)$ . If  $P \cap Q = PQ$  for each  $Q \in X^{(1)}(A)$  such that  $Q \neq P$ , then  $P$  is an invertible ideal of  $A$ .

**Theorem 8.** *Let  $A$  be a Krull ring of Mori-Nagata type. Then the following conditions are equivalent for  $A$ :*

- (1)  $A$  is a locally factorial ring with  $\dim(A) \leq 2$ .
- (2)  $P \cap Q = PQ$  for each incomparable prime ideals  $P, Q$  of  $A$ .
- (3) Each ideal of  $A$  is a product of primary ideals of  $A$ .

*Proof.* (3) implies (1) by [3, Corollary 6]. It is straightforward to see that (1) implies (2). Assume (2). We must prove (3). By Theorem 6 we have  $\dim(A) \leq 2$ . Then Lemma 2 implies that  $A$  is a Noetherian ring. By [2, Corollary (2. 8)], each ideal of  $A$  is a product of primary ideals of  $A$ .

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