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ON REAL HYPERSURFACES OF A COMPLEX SPACE FORM

U-HANG KI* and YOUNG JIN SUH

Introduction. A complex n -dimensional Kaehler manifold of constant holomorphic sectional curvature c is called a complex space form, which is denoted by $M_n(c)$. Let F be its complex structure. The complete and simply connected complex space form consists of a complex projective space CP^n , a complex Euclidean space C^n or a complex hyperbolic space CH^n , according as $c > 0$, $c = 0$ or $c < 0$.

In the study of real hypersurfaces of a complex projective space CP^n , Takagi [11] classified all homogeneous real hypersurfaces of CP^n . He showed also that real hypersurfaces of CP^n with 2 or 3 distinct constant principal curvatures are homogeneous.

On the other hand, Cecil and Ryan [2] studied pseudo-Einstein real hypersurfaces of CP^n on which $\xi = -FC$ is principal, where C is the unit normal vector field on M . They showed that if ξ is principal, then M lies on a tube over a Kaehler submanifold. By making use of this notion and the results of Takagi's classification, Kimura [3] proved the following.

Theorem A. *Let M be a connected real hypersurface of CP^n . Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*

- (A₁) a tube over a hyperplane CP^{n-1} .
- (A₂) a tube over a totally geodesic CP^k ($1 \leq k \leq n-2$).
- (B) a tube over a complex quadric Q_{n-1} .
- (C) a tube over $CP^1 \times CP^{(n-1)/2}$ and $n(\geq 5)$ is odd.
- (D) a tube over a complex Grassmann $G_{2,5}(C)$ and $n = 9$.
- (E) a tube over a Hermitian symmetric space $SO(10)/U(5)$, and $n = 15$.

According to Takagi's classification [11], the principal curvatures and their multiplicities of the above homogeneous real hypersurfaces are given.

On the other hand, real hypersurfaces of a complex hyperbolic space CH^n have also been investigated by Berndt [1], Montiel [8], Montiel and

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Romero [9]. In particular, by using the notion of the tube in Cecil and Ryan [2], Montiel [8] also classified the real hypersurface of complex hyperbolic space with at most two distinct principal curvatures. Recently, Berndt [1] classified all real hypersurfaces with constant principal curvature of CH^n under the condition such that ξ is principal. Namely he proved the following.

Theorem B. *Let M be a connected real hypersurface of $CH^n (n \geq 2)$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*

- (A₁) *a horosphere in CH^n .*
- (A₂) *a tube over CH^k for a $k = 0, 1, \dots, n-1$.*
- (B) *a tube over RH^n .*

For the principal curvatures and their multiplicities of the above hypersurfaces are also given in [1].

The purpose of this paper is to characterize some real hypersurfaces of $M_n(c)$, $c \neq 0$, by using above classification theorems. The authors would like to express their thanks to the referee for his valuable comments.

1. Preliminaries. Let M be a real hypersurface of a complex n dimensional complex space form $M_n(c)$, and let C be a unit normal vector field on a neighborhood of a point x in M . Let us denote by F the almost complex structure of $M_n(c)$. For any local vector field X on a neighborhood of x in M , the transformations of X and C under F can be given by

$$FX = \phi X + \eta(X)C, \quad FC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M , while η and ξ denote a 1-form and a vector field on a neighborhood of X in M respectively. Then it is seen that $g(\xi, X) = \eta(X)$, where g denotes the induced Riemannian metric on M . The set of tensors (ϕ, ξ, η, g) is called an almost contact structure on M . They satisfy the following

$$(1.1) \quad \phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation. Furthermore, the covariant derivatives of the structure tensors are given by

$$(1.2) \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX,$$

where ∇ is the Riemannian connection of g and A denotes the shape operator with respect to the unit normal C on M .

Since the ambient space is of constant holomorphic sectional curvature c , the equations of Gauss and Codazzi are respectively given as follows

$$(1.3) \quad R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ - 2g(\phi X, Y)\phi Z\}/4 + g(AY, Z)AX - g(AX, Z)AY,$$

$$(1.4) \quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ denotes the covariant derivative of the shape operator A with respect to X .

The Ricci tensor S of M is the tensor of type $(0, 2)$ given by $S(X, Y) = \text{tr}\{Z \rightarrow R(Z, X)Y\}$. Also it may be regarded as the tensor of type $(1, 1)$ and denoted by $S : TM \rightarrow TM$; it satisfies $S(X, Y) = g(SX, Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$(1.5) \quad S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 + hA - A^2,$$

where we have put $h = \text{tr}A$. A real hypersurface M of $M_n(c)$ is said to be pseudo-Einstein if the Ricci tensor S satisfies

$$(1.6) \quad SX = aX + b\eta(X)\xi$$

for any vector field X tangent to M and some functions a and b on M .

2. Certain Lemmas. Let M be a real hypersurface of a complex space form $M_n(c)$. The shape operator A of M can be considered as a symmetric $(2n-1, 2n-1)$ -matrix. Now we assume that the structure vector ξ is an eigenvector of A , that is, $A\xi = \alpha\xi$. Then the second formula of (1.2) gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX,$$

from which it follows that

$$(2.1) \quad g((\nabla_X A)Y, \xi) = (X\alpha)\eta(Y) + \alpha g(Y, \phi AX) - g(Y, A\phi AX).$$

By using equation of Codazzi to (2.1) we have

$$(2.2) \quad cg(X, \phi Y)/2 = (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((\phi A + A\phi)X, Y) \\ - 2g(A\phi AX, Y).$$

Putting $X = \xi$ or $Y = \xi$ in (2.2), then we see that $X\alpha = (\xi\alpha)\eta(X)$, or $Y\alpha = (\xi\alpha)\eta(Y)$ and hence (2.2) reduces to

$$(2.3) \quad cg(X, \phi Y)/2 = \alpha g((\phi A + A\phi)X, Y) - 2g(A\phi AX, Y).$$

First of all we prove the following.

Lemma 2.1. *Let M be a real hypersurface of a complex space form $M_n(c)$. If $\phi A + A\phi = 0$, then $c = 0$.*

Proof. By the assumption we have that ξ is an eigenvector of A . From this assumption, and the almost contact structure (ϕ, ξ, η, g) and (2.3) it follows that

$$(2.4) \quad A^2 = -cI/4 + (\alpha^2 + c/4)\eta \otimes \xi.$$

We notice here that the holomorphic sectional curvature c is non-positive. Thus (2.1) and (2.3) imply

$$(2.5) \quad g((\nabla_x A)Y, \xi) = -cg(Y, \phi X)/4 + \alpha g(Y, \phi AX) + (X\alpha)\eta(Y).$$

Differentiating (2.4) covariantly along M , we find

$$(2.6) \quad (\nabla_x A)AY + A((\nabla_x A)Y) = 2\alpha(\xi\alpha)\eta(X)\eta(Y)\xi + (\alpha^2 + c/4) \\ \times \{g(\nabla_x \xi, Y)\xi + \eta(Y)\nabla_x \xi\},$$

from which, taking the skew-symmetric part and using the equation of Codazzi, we have

$$(\nabla_x A)AY - (\nabla_y A)AX = c\alpha g(\phi X, Y)\xi/2 + \alpha^2\{\eta(Y)\phi AX - \eta(X)\phi AY\},$$

where we have used the fact $\phi A + A\phi = 0$. Equivalently it follows that

$$g(AY, (\nabla_z A)X) - g(AZ, (\nabla_y A)X) = c\alpha\eta(X)g(\phi Z, Y)/2 \\ + \alpha^2\{\eta(Y)g(\phi AZ, X) - \eta(Z)g(\phi AY, X)\},$$

from which, also using the equation of Codazzi, we have

$$(2.7) \quad g(AY, (\nabla_x A)Z) - g(AZ, (\nabla_x A)Y) = c\alpha/2\{\eta(X)g(Y, \phi Z) \\ + \eta(Y)g(X, \phi Z) + \eta(Z)g(Y, \phi X)\} \\ + (\alpha^2 - c/4)\{\eta(Y)g(\phi AZ, X) - \eta(Z)g(\phi AY, X)\}.$$

Summing up (2.6) and (2.7), we have

$$(\nabla_x A)AY = \alpha(\xi\alpha)\eta(X)\eta(Y)\xi + \alpha^2\eta(Y)\phi AX + cg(\phi AY, X)\xi/4 \\ + c\alpha\{-\eta(X)\phi Y - \eta(Y)\phi X + g(Y, \phi X)\xi\}/4.$$

From which, substituting AY into Y and using (2.4) and (2.5), we have that

$$(2.8) \quad c(\nabla_x A)Y = c(\xi\alpha)\eta(X)\eta(Y)\xi + c^2\{g(\phi Y, X)\xi - \eta(Y)\phi X\}/4 \\ + c\alpha\{\eta(X)\nabla_y \xi + g(\nabla_y \xi, X)\xi + \eta(Y)\nabla_x \xi\}.$$

Now we take an orthonormal frame $\{E_i\}$ of $T_x(M)$ such that $\nabla_{E_i}E_j = 0$ ($i, j, \dots = 1, 2, \dots, 2n-1$). Differentiating (2.8) with respect to E_i and using

the fact that $E_i \alpha = (\xi \alpha) \eta(E_i)$, it follows from the almost contact structure that we have

$$(2.9) \quad \begin{aligned} \sum_{i,j} c g(\phi E_i, E_j) (\nabla_{E_i} \nabla_{E_j} A) Y &= \frac{c^2}{4} \sum_i \{ g(\phi Y, \phi E_i) \nabla_{E_i} \xi - g(\nabla_{E_i} \xi, Y) \phi^2 E_i \} \\ &+ c \alpha \sum_i \{ g(\nabla_{E_i} \xi, \phi E_i) \nabla_Y \xi + g(\nabla_{E_i} \nabla_Y \xi - \nabla_{\nabla_{E_i} Y} \xi, \phi E_i) \xi \\ &+ g(\nabla_{E_i} \xi, Y) \nabla_{\phi E_i} \xi + g(\nabla_Y \xi, \phi E_i) \nabla_{E_i} \xi + \eta(Y) \nabla_{E_i} \nabla_{\phi E_i} \xi \}. \end{aligned}$$

On the other hand, we have

$$\sum_i g(\nabla_{E_i} \xi, \phi E_i) = \sum_i \{ g(AE_i, E_i) - g(AE_i, \eta(E_i) \xi) \} = 0,$$

where in the last step we have used the fact that the mean curvature of A coincides with α because of the assumption $A\phi + \phi A = 0$. From the almost contact structure and the fact ξ is principal the following formula also vanishes.

$$\sum_i \{ g(\nabla_{E_i} \xi, Y) \nabla_{\phi E_i} \xi + g(\nabla_Y \xi, \phi E_i) \nabla_{E_i} \xi \} = -\nabla_{AY} \xi + \nabla_{AY} \xi = 0.$$

If we use the formula $c \sum_i (\nabla_{E_i} A) E_i = c(\xi \alpha) \xi$ and $c g((\nabla_{E_i} A) Y, \xi) = c(\xi \alpha) \cdot \eta(Y)$ which come from (2.8), then we get

$$c \sum_i g(\nabla_{E_i} \nabla_Y \xi - \nabla_{\nabla_{E_i} Y} \xi, \phi E_i) = c \sum_i \{ g(Y, (\nabla_{E_i} A) E_i) - g((\nabla_{E_i} A) Y, \xi) \} = 0.$$

Also using the formula $c \sum_i (\nabla_{E_i} A) E_i = c(\xi \alpha) \xi$, we have

$$c \sum_i \nabla_{E_i} \nabla_{\phi E_i} \xi = c \sum_i \{ (\nabla_{E_i} A) E_i - g(\xi, (\nabla_{E_i} A) E_i) \xi \} = 0.$$

Thus from these equations we see that (2.9) reduces to the following

$$(2.10) \quad \begin{aligned} \sum_{i,j} c g(\phi E_i, E_j) (\nabla_{E_i} \nabla_{E_j} A) Y &= \frac{c^2}{4} \sum_i \{ g(\phi Y, \phi E_i) \nabla_{E_i} \xi - g(\nabla_{E_i} \xi, Y) \phi^2 E_i \} \\ &= \frac{c^2}{4} \{ \nabla_Y \xi + \sum_i g(\nabla_{E_i} \xi, Y) E_i \} = \frac{c^2}{2} \phi AY, \end{aligned}$$

where in the last equality we have used the assumption $A\phi + \phi A = 0$.

If we use the Ricci-formula to (2.10) for the shape operator A , then we get

$$(2.11) \quad c \sum_{i,j} g(\phi E_i, E_j) \{ R(E_i, E_j) AY - A(R(E_i, E_j) Y) \} = c^2 \phi AY.$$

On the other hand, from the equations of Gauss and the assumption $A\phi + \phi A = 0$ it follows that

$$\begin{aligned} \sum_{i,j} g(\phi E_i, E_j) R(E_i, E_j) Y &= \frac{c}{4} \sum_i \{ g(\phi E_i, Y) E_i - g(E_i, Y) \phi E_i \\ &\quad + g(\phi^2 E_i, Y) \phi E_i - g(\phi E_i, Y) \phi^2 E_i - 2g(\phi E_i, \phi E_i) \phi Y \} \\ &\quad + \sum_i \{ g(A\phi E_i, Y) A E_i - g(A E_i, Y) A \phi E_i \} = -cn\phi Y + 2A^2 \phi Y, \end{aligned}$$

from which together with (2.4), (2.11) reduces to

$$c^2 \phi A Y = 0.$$

If $c \neq 0$, then $\phi A Y = 0$. It follows from the almost contact structure that we have $A Y = \alpha \eta(Y) \xi$. The rank of A at a point x in M is called the type number and is denoted by $t(x)$. Thus it means that the type number $t(x)$ of any point x in M is at most 1. It is however seen that (cf. Yano and Kon [13]) $t(x) > 1$ at some point x of M for $c \neq 0$. So it is contradiction. Hence we have $c = 0$. This completes the above proof.

From Lemma 2.1. we have the following.

Proposition 2.2. *Let M be a real hypersurface of a complex space form $M_n(c)$. If $\phi A + A\phi = 0$, then M is cylindrical.*

Proof. From the assumption it follows that $A\xi = \alpha\xi$. Since $c = 0$ by Lemma 2.1, (2.3) implies $A\phi A = 0$, from which it follows that $(\phi A)^2 = \phi A\phi A = -\phi A A\phi = \phi A^t(\phi A)$. Thus $\text{tr}(\phi A)^t(\phi A) = 0$, that is, $\phi A = 0$. Then $A X = \alpha \eta(X) \xi$. Hence M is cylindrical.

Also by using Lemma 2.1 we get the following.

Lemma 2.3. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. If ξ is an eigenvector of A , then $\alpha = \eta(A\xi)$ is locally constant.*

Proof. Since $X\alpha = \beta \eta(X)$, we have $\nabla_X \text{grad } \alpha = (X\beta) \xi + \beta \nabla_X \xi$, where we have put $\beta = \xi \alpha$. From which together with the fact $g(\nabla_X \text{grad } \alpha, Y) = g(\nabla_Y \text{grad } \alpha, X)$ it follows that

$$(2.12) \quad (X\beta) \eta(Y) - (Y\beta) \eta(X) + \beta g((\phi A + A\phi) X, Y) = 0.$$

Putting $X = \xi$ or $Y = \xi$ in (2.12), we get $X\beta = (\xi\beta) \eta(X)$ or $Y\beta = (\xi\beta) \eta(Y)$. Thus (2.12) reduces to

$$\beta g((\phi A + A\phi)X, Y) = 0.$$

By Lemma 2.1 there are no points on M at which $\phi A + A\phi = 0$, which yields that $\beta = 0$ on M . This means α is constant on M .

Remark. For a real hypersurface of a complex projective space CP^n Maeda proved that α is constant ([7]).

3. Real hypersurfaces of CH^n satisfying certain commutative condition.

A characterization of the class of hypersurfaces with more than 3 distinct principal curvatures of CP^n is studied by Kimura [4], who proves the following.

Theorem C. *Let M be a real hypersurface of $CP^n (n \geq 3)$. Then M satisfies $S\phi = \phi S$ if and only if M lies on a tube of radius r over one of the following Kaehler submanifolds;*

- (A) *a totally geodesic CP^k , ($1 \leq k \leq n-1$), where $0 < r < \pi/2$,*
- (B) *a complex quadric Q^{n-1} , where $0 < r < \pi/4$ and $\cot^2 2r = n-2$,*
- (C) *$CP^1 \times CP^{(n-1)/2}$, where $0 < r < \pi/4$, $\cot^2 2r = 1/(n-2)$ and $n(\geq 5)$ is odd,*
- (D) *complex Grassmann $G_{2,5}(C)$, where $0 < r < \pi/4$, $\cot^2 2r = 3/5$ and $n = 9$,*
- (E) *Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \pi/4$, $\cot^2 2r = 5/9$ and $n = 15$.*

This section is devoted to the investigation about certain real hypersurfaces of CH^n under the condition such that the Ricci tensor and the structure tensor are commutative. Now we introduce the following.

Lemma 3.1. *Let M be a real hypersurface of $CH^n (n \geq 3)$ and $P = A^2 - fA$ such that f is a smooth function on M . If M satisfies the condition*

$$(3.1) \quad P\phi = \phi P,$$

then ξ is a principal vector at each point of M .

For the real hypersurface of $CP^n (n \geq 3)$ Kimura [4] proved that ξ is principal under the condition (3.1) by using Cecil-Ryan's method in the paper [2]. If we use the same method as used in [4], we can obtain the above Lemma. Thus we omit the proof of the Lemma 3.1.

By the above Lemma and Lemma 2.3 we get the following

Lemma 3.2. *Let M be a real hypersurface of $CH^n (n \geq 3)$ satisfying $S\phi = \phi S$. Then the principal curvature α corresponding to ξ is locally constant.*

By Lemma 3.1 we have (2.3). Thus for the complex hyperbolic space CH^n (2.3) implies that $2\phi + 2A\phi A = \alpha(A\phi + \phi A)$. From which, for a unit vector X orthogonal to ξ such that $AX = \lambda X$ we get

$$(3.3) \quad (2\lambda - \alpha)A\phi X = (\alpha\lambda - 2)\phi X.$$

Let V be an open set consisting of points x of M at which $(2\lambda - \alpha)_x \neq 0$. Then $A\phi X = \mu\phi X$ on V , where we have put $\mu = (\alpha\lambda - 2)/(2\lambda - \alpha)$. From (1.5) and (3.1) it follows that

$$(3.4) \quad (\mu - \lambda)|(\mu + \lambda) - h| = 0,$$

where h means the trace of A . Thus $\mu = \lambda$ or $h = \lambda + \mu$ holds on V .

For the case $\mu = \lambda$, it is a root of a quadratic equation $x^2 - \alpha x + 1 = 0$ with constant coefficients, which means that λ is constant. Since $\alpha^2 \geq 4$, we put $\alpha = \pm 2$ or $\alpha = 2 \coth 2\theta$. Then it is seen that $\lambda = \pm 1$ for $\alpha = \pm 2$ and $\lambda = \coth \theta$ or $\tanh \theta$ for $\alpha = 2 \coth 2\theta$. This means that we have at most of five kinds of principal curvatures α , $\coth \theta$, $\tanh \theta$, and λ, μ such that $\lambda + \mu = h$. Since (3.3) implies that multiplicities of λ and μ are equal, say m_1 , we can put

$$h = (\lambda + \mu)m_1 + m_2 \coth \theta + m_3 \tanh \theta + \alpha,$$

from which together with $h = \lambda + \mu$ it follows that

$$(3.5) \quad (1 - m_1)h = m_2 \coth \theta + m_3 \tanh \theta + \alpha.$$

Since the right hand side of (3.5) is positive or negative according as $\theta > 0$ or $\theta < 0$, respectively, we have $m_1 \neq 1$, and h is constant.

On the other hand, it follows from $h = \lambda + \mu$ that $2\lambda^2 - 2h\lambda + \alpha h - 2 = 0$. Thus all principal curvatures are constant on V . Since V is open and the constancy of principal curvatures gives that V is closed, V coincides with M itself or it is empty. If V is empty, then $2\lambda = \alpha$ gives $\alpha\lambda = 2$ because of (3.3). Thus $\lambda = \pm 1$. Together with this fact we conclude that all principal curvatures are constant on M . Thus we have the following.

Theorem 3.3. *Let M be a real hypersurface of $CH^n (n \geq 3)$. Then the Ricci tensor of M commutes with the almost contact structure of M induced from CH^n if and only if M is of type A_1, A_2 .*

Proof. By the classification Theorem of Berndt M is of type A_1, A_2 or B .

On the other hand, Montiel and Romero show that the real hypersurface M of CH^n is of type A_1, A_2 if and only if the almost contact structure tensor commutes with the second fundamental form. Hence the type A_1, A_2 naturally satisfy $S\phi = \phi S$.

Now we suppose that M is of B -type. Then the table of Berndt [1] gives that $\alpha = 2 \tanh 2\theta$, $\lambda = \tanh \theta$ and $\mu = \coth \theta$. Since multiplicities of λ and μ are equal, (3.4) gives $h = \alpha + (n-1)(\lambda + \mu) = \alpha + (n-1)h$. Thus $(n-2)h + \alpha = 0$, from which together with the fact that $\alpha = 4\lambda/(1+\lambda^2)$ it follows that $4\lambda^2 + (n-2)(1+\lambda^2)^2 = 0$. This contradicts.

Remark. For the real hypersurface of $CP^n (n \geq 3)$ Kimura [4] proved that $S\phi = \phi S$ if and only if M is of type A_1, A_2 , or M is locally congruent to one of a certain hypersurface of type B, C, D or E .

4. Real hypersurfaces of $M_n(c)$, $c \neq 0$. Let M be a pseudo-Einstein real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. Then the Ricci tensor S of M is given by $SX = aX + b\eta(X)\xi$ where a and b are C^∞ -functions. From which it naturally satisfies the following.

$$(4.1) \quad A\xi = \alpha\xi,$$

$$(4.2) \quad R(X, Y)(SZ) + R(Y, Z)(SX) + R(Z, X)(SY) = 0,$$

for any X, Y , and Z in $\xi^\perp$, where we have put $\xi^\perp$ the orthogonal complement of ξ in $T_x(M)$ for any x in M .

In this section, we are concerned with the converse problem. Namely we will give another characterization of pseudo-Einstein real hypersurfaces of $M_n(c)$, $c \neq 0$, with (4.1) and (4.2). From (4.1) it follows that $\eta(AX) = 0$ for any X in $\xi^\perp$. By taking account of (1.3) and (1.5), the above equation (4.2) is equivalent to

$$(4.3) \quad g(QZ, Y)\phi X + g(QX, Z)\phi Y + g(QY, X)\phi Z \\ + 2g(\phi Y, Z)\phi PX + 2g(\phi Z, X)\phi PY + 2g(\phi X, Y)\phi PZ = 0$$

for any X, Y and Z in $\xi^\perp$, where we have put $P = A^2 - hA$, $h = \text{tr}A$, and $Q = P\phi + \phi P$. Since ϕ is non-degenerate on $\xi^\perp$, (4.3) reduces to

$$(4.4) \quad g(QZ, Y)X + g(QX, Z)Y + g(QX, Y)Z \\ + 2\{g(\phi Y, Z)PX + g(\phi Z, X)PY + g(\phi X, Y)PZ\} = 0.$$

For a symmetric transformation $P = A^2 - hA$ let X , Y , and Z be orthonormal eigenvectors such that

$$(4.5) \quad PX = \alpha_r X, PY = \alpha_s Y, \text{ and } PZ = \alpha_t Z.$$

Thus, from which together with (4.4) it follows that

$$(4.6) \quad \begin{aligned} g(QZ, Y) - 2\alpha_r g(\phi Z, Y) &= 0, \\ g(QX, Z) - 2\alpha_s g(\phi X, Z) &= 0, \\ g(QY, X) - 2\alpha_t g(\phi Y, Z) &= 0. \end{aligned}$$

Using (4.5) again to (4.6), we have

$$(4.7) \quad \begin{aligned} (\alpha_s + \alpha_t - 2\alpha_r)g(\phi Y, Z) &= 0, \\ (\alpha_r + \alpha_t - 2\alpha_s)g(\phi X, Z) &= 0, \\ (\alpha_r + \alpha_s - 2\alpha_t)g(\phi Y, X) &= 0. \end{aligned}$$

Let us now decompose $T_x(M)$ as following: $T_x(M) = P(\alpha_1) \oplus \cdots \oplus P(\alpha_p)$, where $P(\alpha_r) = \{X \in T_x(M) \mid PX = \alpha_r X\}$ ($r = 1, \dots, p$), $\alpha_1, \dots, \alpha_p$ are all distinct, and ξ in $P(\alpha_1)$.

Lemma 4.1. *If $p \geq 2$ and $\dim P(\alpha_1) \geq 2$, then $\dim P(\alpha_1) = 2$, and $\dim P(\alpha_r) = 1$ ($r \geq 2$).*

Proof. Suppose $\dim P(\alpha_1) \geq 3$ or $\dim P(\alpha_r) \geq 2$ for some $r \geq 2$. Then for any $s \leq p$, $s \neq r$, and any linearly independent vectors X, Y in $P(\alpha_r)$ ($r = 1, \dots, p$), and Z in $P(\alpha_s)$, (4.7) give rise to

$$(4.8) \quad \begin{aligned} (\alpha_s - \alpha_r)g(\phi Y, Z) &= 0, \\ (\alpha_s - \alpha_r)g(\phi X, Z) &= 0, \\ (\alpha_r - \alpha_s)g(\phi Y, X) &= 0. \end{aligned}$$

Since $\alpha_r \neq \alpha_s$, $g(\phi Y, Z) = g(\phi X, Z) = g(\phi Y, X) = 0$, from which it follows that ϕX is orthogonal to $P(\alpha_s)$ for any s different from r . Thus ϕX is contained in $P(\alpha_r)$. In particular, if we put $Y = X$, then $g(\phi X, Y) = g(\phi X, \phi X) \neq 0$. This contradicts. Thus we have the above Lemma.

Lemma 4.2. *If $p \geq 2$, then $\dim P(\alpha_1) = 1$.*

Proof. If we suppose $\dim P(\alpha_1) \neq 1$, then by Lemma 1, we get $\dim P(\alpha_1) = 2$. Thus we can take a vector X in $P(\alpha_1)$ orthogonal to ξ . Since $\dim P(\alpha_1) = 2$, ϕX is not contained in $P(\alpha_1)$. Thus ϕX is in $P(\alpha_2) \oplus \cdots \oplus P(\alpha_p)$. Hence we can assume that there exists an element Y in $P(\alpha_2)$ such

that $g(\phi X, Y) \neq 0$.

Now let $p \geq 3$. Then let us take X, Y , and Z be orthonormal vectors in $P(\alpha_1), P(\alpha_2)$, and $P(\alpha_r)$ ($r \geq 3$), respectively. From which and (4.7) it follows that $(\alpha_1 + \alpha_2 - 2\alpha_r)g(\phi X, Y) = 0$. Thus, we get $2\alpha_r = \alpha_1 + \alpha_2$ for $r \geq 3$ because of $g(\phi X, Y) \neq 0$. Hence we have $p = 3$. This implies that $\dim P(\alpha_1) + \dim P(\alpha_2) + \dim P(\alpha_3) = 4$ by virtue of Lemma 1. This contradicts the fact $\dim T_x(M) \geq 5$ for $n \geq 3$. Thus we should have $p = 2$. But in this case we also have $\dim P(\alpha_1) + \dim P(\alpha_2) = 3$ by Lemma 1. This also makes contradiction. Thus we get the above Lemma.

Lemma 4.3. $p = 2$.

Proof. Firstly we now consider for the case $p \geq 3$. Then by Lemma 4.2. $\dim P(\alpha_1) = 1$. And we will show $\dim P(\alpha_r) = 1$ ($r \geq 2$) for $p \geq 3$. Thus, if we suppose $\dim P(\alpha_r) \geq 2$ for some $r \geq 2$, then for any linearly independent vectors X, Y in $P(\alpha_r)$ and Z in $P(\alpha_s)$, $r \neq s$, $s \geq 2$, we get $g(\phi X, Y) = g(\phi Y, Z) = g(\phi Z, X) = 0$ by virtue of (4.8). Hence we evoke the same contradiction as Lemma 4.1. Thus we have $\dim P(\alpha_r) = 1$ for any $r \geq 2$.

Now we consider for $p \geq 4$. Then from above facts $\dim P(\alpha_r) = 1$ for any $r \geq 2$. Thus for X in $P(\alpha_2)$, ϕX is contained in $P(\alpha_3) \oplus \cdots \oplus P(\alpha_p)$. Hence we can take an element Y in $P(\alpha_3)$ such that $g(\phi X, Y) \neq 0$. For Z in $P(\alpha_r)$, $r \geq 4$, we have

$$(\alpha_2 + \alpha_3 - 2\alpha_r)g(\phi X, Y) = 0.$$

Since $g(\phi X, Y) \neq 0$, we get $2\alpha_r = \alpha_2 + \alpha_3$ for $r \geq 4$. Thus $p = 4$. This implies $\sum_{r=1}^4 \dim P(\alpha_r) = 4$. This contradicts. Hence $p = 3$. For this case we can also have $\sum_{r=1}^3 \dim P(\alpha_r) = 3$. This also makes contradiction. Thus we should have $p = 2$.

From Lemmas 4.1, 4.2 and 4.3 we get the following.

Theorem 4.4. *Let M be a real hypersurface of a complex space form $M_n(c)$, $c \neq 0$. If M satisfies (4.1) and (4.2), then M is pseudo-Einstein.*

Proof. By Lemma 4.3 we have $\dim P(\alpha_1) = 1$, and $\dim P(\alpha_2) = 2n - 2$. Thus

$$P = \begin{pmatrix} \alpha_1 & & & 0 \\ & \alpha_2 & & \\ & & \ddots & \\ 0 & & & \alpha_2 \end{pmatrix}$$

This gives $P = \alpha_2 I + (\alpha_1 - \alpha_2)\eta \otimes \xi$. From which and (1.5) it follows that $S = \{(2n+1)c/4 - \alpha_2\}I + (\alpha_2 - \alpha_1 - 3c)\eta \otimes \xi$. Hence M is pseudo-Einstein.

Remark. Recently Kimura and Maeda [5] introduced the notion of η -parallel second fundamental form A , that is, $g((\nabla_X A)Y, Z) = 0$ for any X, Y , and Z in $\xi^\perp$. And they showed that any real hypersurface M of CP^n with η -parallel second fundamental form A and principal vector ξ is of type A_1, A_2 and B .

The condition ξ is principal can not be omitted because a ruled real hypersurface M in CP^n has η -parallel second fundamental form A but ξ is not principal.

5. Real hypersurfaces of CP^n satisfying certain conditions. To give another characterization of some type of real hypersurfaces of the complex projective space CP^n we now introduce the following.

Lemma 5.1. (Takagi [12]) *If M is a connected complete totally η -umbilical real hypersurface in $CP^n (n \geq 2)$, then M is of type A_1 .*

Lemma 5.2. (Yano and Kon [14]) *Let M be a connected complete real hypersurface in $CP^n (n \geq 3)$. If $\phi A + A\phi = k\phi$ for some constant $k \neq 0$, then M is of type A_1 or B .*

By above Lemmas we can see that the type A_1 or B satisfies the condition

$$(*) \quad S\phi + \phi S = k_1 \phi \quad (k_1 : \text{constant}).$$

And also pseudo-Einstein real hypersurfaces of CP^n satisfy $(*)$. As the converse problem in this section we are devoted to the investigation of the real hypersurfaces of CP^n satisfying $(*)$ and with principal structure vector field ξ .

By (1.5), $(*)$ is equivalent to

$$(5.1) \quad A^2 \phi + \phi A^2 - h(A\phi + \phi A) = k\phi,$$

where we have put $k = 2(2n+1) - k_1$, and h means the trace of A .

Since CP^n has the Fubini-Study metric and the constant holomorphic sectional curvature $c = 4$, (2.3) implies that

$$(5.2) \quad \alpha(\phi A + A\phi) - 2A\phi A + 2\phi = 0.$$

From which it follows that if X is an eigenvector of A with eigenvalue λ and if X is orthogonal to ξ , then ϕX is an eigenvector of A with eigenvalue $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$. With this fact (5.1) implies

$$(5.3) \quad \mu^2 + \lambda^2 - h(\mu + \lambda) = k.$$

Substituting $\mu = (\alpha\lambda + 2)/(2\lambda - \alpha)$ into (5.3), we get

$$(5.4) \quad 4\lambda^4 - 4(\alpha + h)\lambda^3 + 2(\alpha^2 + h\alpha - 2k)\lambda^2 + 4(\alpha - h + k\alpha)\lambda + 4 + 2\alpha h - \alpha^2 k = 0.$$

Let $\lambda_1, \lambda_2, \lambda_3$ and λ_4 be the roots of the above equation. Then from the roots and coefficient of (5.4) it follows that

$$(5.5) \quad \begin{cases} \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = \alpha + h, \\ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_1 \lambda_4 + \lambda_2 \lambda_3 + \lambda_2 \lambda_4 + \lambda_3 \lambda_4 = (\alpha^2 + h\alpha - 2k)/2, \\ \lambda_1 \lambda_2 \lambda_3 + \lambda_1 \lambda_3 \lambda_4 + \lambda_2 \lambda_3 \lambda_4 + \lambda_1 \lambda_2 \lambda_4 = -(\alpha - h + k\alpha), \\ \lambda_1 \lambda_2 \lambda_3 \lambda_4 = (4 + 2\alpha h - \alpha^2 k)/4. \end{cases}$$

Substituting $h = \alpha + m_1 \lambda_1 + m_2 \lambda_2 + m_1(\alpha\lambda_1 + 2)/(2\lambda_1 - \alpha) + m_2(\alpha\lambda_2 + 2)/(2\lambda_2 - \alpha)$ into the above equation, and noticing α and k are constant, we can see that (5.5) consists of 4 linearly independent equation, where m_j denotes the (constant) multiplicities of principal curvatures ($j = 1, 2$). Thus M has at most five distinct constant principal curvatures. Hence by Theorem A, M is homogeneous. Then by Takagi's classification of homogeneous real hypersurfaces we can suppose M is of type A_1, A_2, B, C, D , and E .

Firstly, we suppose that M is one of type B, C, D and E . Then from the table given in [11] its type has the following roots: $\lambda = \cot(r - \pi/4)$, $\mu = -\tan(r - \pi/4)$, and $\alpha = 2 \cot 2r$. Hence $\lambda + \mu = -4/\alpha$, $\lambda\mu = -1$. Thus, by (5.3) we get $k = (4/\alpha)^2 + h(4/\alpha) + 2$. From which, (5.4) can be rewritten as following

$$(5.6) \quad 2\alpha^2 \lambda^4 - 2\alpha^2(\alpha + h)\lambda^3 + |\alpha^4 + h\alpha^3 - 4\alpha^2 - 8h\alpha - 32|\lambda^2 + 2(3\alpha^3 + 3h\alpha^2 + 16\alpha)\lambda - \alpha^2(\alpha^2 + \alpha h + 6) = 0$$

Then (5.6) can be decomposed into

$$(5.7) \quad (\alpha\lambda^2 + 4\lambda - \alpha)(2\alpha\lambda^2 - 2(\alpha^2 + h\alpha + 4)\lambda + (\alpha^3 + h\alpha^2 + 6\alpha)) = 0.$$

Since $\cot(r - \pi/4)$, $-\tan(r - \pi/4)$ satisfy $\alpha\lambda^2 + 4\lambda - \alpha = 0$, another roots $\cot r$, $-\tan r$ of C , D , and E should satisfy

$$(5.8) \quad 2\alpha\lambda^2 - 2(\alpha^2 + h\alpha + 4)\lambda + \alpha^3 + h\alpha^2 + 6\alpha = 0.$$

On the other hand, $\cot r$, $-\tan r$ are roots of $\lambda^2 - \alpha\lambda - 1 = 0$. From this fact and the root and coefficient of (5.8), it follows that

$$h\alpha + 4 = 0, \text{ and } \alpha^2 + h\alpha + 8 = 0.$$

Thus $\alpha^2 + 4 = 0$. This contradicts. Thus the type of C , D , and E can not occur.

Next we consider for the type A_1 , A_2 . Then we introduce the following.

Lemma 5.3. (Okumura [10]) *Let M be a real hypersurface of CP^n . Then M is of type A_1 or A_2 if and only if $A\phi = \phi A$.*

Since by Lemma 5.1. the type A_1 naturally satisfies (*) and its structure vector ξ is principal, we restrict our attention to the type A_2 . Then using Lemma 5.3 to (5.1), we get

$$(5.9) \quad A^2\phi - hA\phi = k\phi/2.$$

From the table of type A_2 given in [11] it follows that for an eigenvector X such that $AX = -\tan r X$

$$(5.10) \quad 2 \cot^2 r - 2h \cot r = k.$$

Also for the case $AX = \cot r X$, $A\phi X = -\tan r \phi X$ we get

$$(5.11) \quad 2 \tan^2 r + 2h \tan r = k.$$

From (5.10) and (5.11) it follows that $(\cot r + \tan r)(\cot r - \tan r - h) = 0$. Since $\cot r + \tan r \neq 0$, $h = \cot r - \tan r = \alpha$. Thus $k = 2$. Then (*) implies $S\phi + \phi S = 4n\phi$. From which and Lemma 5.3, it follows $S\phi = \phi S = 2n\phi$. Hence $S = 2nI - 2\eta \otimes \xi$. Thus the type of A_2 satisfying (*) is pseudo-Einstein and M is $M(2n-1, m, (m-1)/(n-m))$ (cf. Yano and Kon [13]). Hence we have the following.

Theorem 5.4. *Let M be a connected complete real hypersurface of CP^n and assume that ξ is principal vector field on M . If M satisfies (*), then M is of type A_1 , B or M is locally congruent to one of a certain hypersurface of type A_2 .*

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