

Mathematical Journal of Okayama University

Volume 42, Issue 1

2000

Article 1

JANUARY 2000

A Characterization of One Dimensional N-Graded Gorenstein Rings of Finite Cohen-Macaulay Representation Type

Tokuji Araya*

*Okayama University

Copyright ©2000 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

Math. J. Okayama Univ. **42** (2000), 61-66

**A CHARACTERIZATION OF ONE DIMENSIONAL
 $\mathbf{N}$ -GRADED GORENSTEIN RINGS OF FINITE
 COHEN-MACAULAY REPRESENTATION TYPE**

TOKUJI ARAYA

1. INTRODUCTION

Let $R = \bigoplus R_n$ be an $\mathbf{N}$ -graded Cohen-Macaulay ring where $R_0 = k$ is a field. We denote by $\text{mod}R$ the category of finitely generated graded R -modules whose morphisms are graded R -homomorphisms that preserve degrees. We also denote by $\text{CM}R$ the full subcategory of $\text{mod}R$ consisting of all graded maximal Cohen-Macaulay modules. In the paper [1], we have shown that if k is an algebraically closed field of characteristic 0 and if R is a one dimensional $\mathbf{N}$ -graded Gorenstein ring of finite Cohen-Macaulay representation type, then there exists an MCM generating exceptional sequence. In that work, we had to compute the dimension of $\text{Ext}_R^n(X, Y)$ as k -vector space, for all indecomposable graded maximal Cohen-Macaulay modules X and Y and for all $n \in \mathbf{N}$. Through this computation, we noticed the importance of the invariants $d(R)$ and $d_n(R)$ of R that are defined as follows:

$$d(R) := \sup\left\{\sum_{n \geq 0} \dim_k \text{Ext}_R^n(X, Y) \mid X, Y \in \text{CM}R \text{ are indecomposable}\right\},$$

$$d_n(R) := \sup\{\dim_k \text{Ext}_R^n(X, Y) \mid X, Y \in \text{CM}R \text{ are indecomposable}\}.$$

In the present paper, we will give a characterization of one dimensional $\mathbf{N}$ -graded Gorenstein rings of finite Cohen-Macaulay representation type utilizing $d(R)$ and $d_0(R)$. More precisely, let R be a positively dimensional $\mathbf{N}$ -graded Gorenstein ring with isolated singularity where $R_0 = k$ is an algebraically closed field of characteristic 0. Then the invariant $d(R)$ can take only 7 values in $\{1, 2, 3, 4, 6, 9, \infty\}$. Moreover, if $d(R) < \infty$, then $\dim R = 1$ and R is isomorphic to one of the rings in the list (1) below and in each case we are able to compute $d(R)$ and $d_n(R)$.

	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$d(R) = d_0(R)$	1	2	3	4	4	6	9
$d_n(R) \ (n \geq 1)$	1	2	1	2	3	4	6

Acknowledgment : The author thanks to Professor Yuji Yoshino for his great help in writing this paper.

2. PRELIMINARIES

In this section, we assume $R = \bigoplus R_n$ is a one dimensional $\mathbf{N}$ -graded Gorenstein ring of finite Cohen-Macaulay representation type and assume that $R_0 = k$ is an algebraically closed field of characteristic 0. In this case, it is known that R is isomorphic to one of the following rings (c.f.[4]).

$$(1) \quad \begin{array}{ll} (A_n) & R = k[x, y]/(y^2 - x^n) \quad (n \geq 2) \\ (D_n) & R = k[x, y]/(xy^2 - x^n) \quad (n \geq 3) \\ (E_6) & R = k[x, y]/(x^3 + y^4) \\ (E_7) & R = k[x, y]/(x^3 + xy^3) \\ (E_8) & R = k[x, y]/(x^3 + y^5) \end{array}$$

Moreover the Auslander-Reiten quiver of CMR for each type can be described as they are shown in [1, Figures (1) – (7)] . We denote by Γ the Auslander-Reiten quiver of CMR.

For indecomposable graded maximal Cohen-Macaulay modules X and Y , we write $X \preceq Y$ if $X \cong Y$ or if there exists a finite path from X to Y in Γ .

Lemma 2.1. [1, Lemma 3.3.] *The following hold for indecomposable graded maximal Cohen-Macaulay modules X and Y .*

- (i) *There are no cyclic paths in Γ .*
- (ii) *If $\text{Hom}(X, Y) \neq 0$, then $X \preceq Y$.*
- (iii) *If $\text{Ext}_R^1(X, Y) \neq 0$, then $Y \preceq \tau X$. Here, τX denotes the Auslander-Reiten translation of X .*

It follows from lemma 2.1.(iii) that, for a fixed X , τX is the right bound of the set $\{Y \in \text{CMR} \mid \text{indecomposable, } \text{Ext}_R^1(X, Y) \neq 0\}$ in Γ . Now, we are giving the left bound of this set.

Lemma 2.2. *For indecomposable graded maximal Cohen-Macaulay modules X and Y , if $\text{Ext}_R^1(X, Y) \neq 0$ then we have $\Omega X \preceq Y \preceq \tau X$. Here, ΩX denotes the first syzygy module of X .*

Proof. Let $0 \rightarrow Y \rightarrow Z \xrightarrow{\pi} X \rightarrow 0$ be a non-split exact sequence. Taking the first syzygy of X ; $0 \rightarrow \Omega X \rightarrow F \rightarrow X \rightarrow 0$ where F is free, we have the commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \Omega X & \longrightarrow & F & \longrightarrow & X \longrightarrow 0 \\ & & f \downarrow & & g \downarrow & & \parallel \\ 0 & \longrightarrow & Y & \longrightarrow & Z & \xrightarrow{\pi} & X \longrightarrow 0 \end{array}$$

Suppose $f = 0$ in this diagram. Then the morphism g will induce a morphism $X \rightarrow Z$ which contradicts the fact that π is not a split epimorphism. Therefore $f \neq 0$, and we get $\Omega X \preceq Y$. $\square$

Lemma 2.3. *For any indecomposable graded maximal Cohen-Macaulay modules X and Y , we have $\sharp\{n \in \mathbf{N} \mid \text{Ext}_R^n(X, Y) \neq 0\} \leq 1$.*

Proof. If X is free, then the lemma is obviously true. Thus we may assume that X is non-free, and hence $\Omega^i X$ ($i > 0$) are also non-free and $\tau \Omega^i X$ ($i \geq 0$) are well-defined. Now assume that $\text{Ext}_R^n(X, Y) \neq 0$ for some $n > 0$. Since $\text{Ext}_R^1(\Omega^{n-1} X, Y) \cong \text{Ext}_R^n(X, Y) \neq 0$, we have $\Omega^n X \preceq Y \preceq \tau \Omega^{n-1} X$ by lemma 2.2. On the other hand, since there exists a sequence $\cdots \prec \Omega^{i+1} X \preceq \tau \Omega^i X \prec \Omega^i X \preceq \tau \Omega^{i-1} X \prec \Omega^{i-1} X \preceq \tau \Omega^{i-2} X \prec \cdots \preceq \tau \Omega X \prec \Omega X \preceq \tau X \prec X$ and since there is no cyclic path in Γ , one sees that $Y \not\preceq \tau \Omega^m X$ for all $m \geq n$ and $\Omega^m X \not\preceq Y$ for all $0 \leq m < n$. Therefore we have $\text{Ext}_R^m(X, Y) = \text{Ext}_R^1(\Omega^{m-1} X, Y) = 0$ for all $m \neq n$ by lemma 2.2. $\square$

3. MAIN THEOREM

In this section, we define the invariants $d(R)$ and $d_0(R)$ by which we will give a characterization of one dimensional $\mathbf{N}$ -graded Gorenstein rings of finite Cohen-Macaulay representation type.

Definition 3.1. For an $\mathbf{N}$ -graded Cohen-Macaulay ring R (not necessarily of dimension one) with $R_0 = k$ being a field, we define $d(R)$ and $d_n(R)$ as follow:

$$d(R) := \sup\left\{\sum_{n \geq 0} \dim_k \text{Ext}_R^n(X, Y) \mid X, Y \in \text{CMR are indecomposable}\right\},$$

$$d_n(R) := \sup\{\dim_k \text{Ext}_R^n(X, Y) \mid X, Y \in \text{CMR are indecomposable}\}.$$

Now we are ready to state our main theorem of this paper.

Theorem 3.2. *Let k be an algebraically closed field of characteristic 0 and let R be a positively dimensional $\mathbf{N}$ -graded Gorenstein ring with isolated singularity where $R_0 = k$. Then the following conditions are equivalent.*

- (i) R is a one dimensional $\mathbf{N}$ -graded Gorenstein ring of finite Cohen-Macaulay representation type.
- (ii) $d(R) < \infty$
- (ii') $d_0(R) < \infty$
- (iii) $d(R) \leq 9$
- (iii') $d_0(R) \leq 9$

To show this theorem, we need the graded version of Brauer-Thrall 1 theorem for graded maximal Cohen-Macaulay modules, due to [4], [3] and [2].

Theorem 3.3 (graded version of Brauer-Thrall 1 theorem). *Let R be an $\mathbf{N}$ -graded Cohen-Macaulay ring with isolated singularity where $R_0 = k$ is a perfect field. If $\sup\{e(X) \mid X \in \text{CM}R \text{ is indecomposable}\} < \infty$, then R is of finite Cohen-Macaulay representation type. Here $e(X)$ denotes the multiplicity of the irrelevant maximal ideal along X .*

Proof of 3.2. The implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (ii') and (iii) $\Rightarrow$ (iii') $\Rightarrow$ (ii') are trivial. First, we show (ii') $\Rightarrow$ (i). Since $d_0(R) < \infty$, we see that $\dim_k R_n = \dim_k \text{Hom}(R, R(n)) \leq d_0(R) < \infty$ for all n . Therefore the Hilbert polynomial of R is constant. Hence $\dim R = 1$. For any indecomposable graded maximal Cohen-Macaulay module X , $\dim_k X_n = \dim_k \text{Hom}(R, X(n)) \leq d_0(R)$ for all n . Therefore the multiplicity $e(X)$ of X is bounded by $d_0(R)$. Hence R is of finite Cohen-Macaulay representation type by theorem 3.3.

To prove (i) $\Rightarrow$ (iii'), it is enough to compute $\sup\{\dim_k \text{Hom}(R, Y), \dim_k \text{Hom}(Y, R), \dim_k \text{Hom}(X_i, Y), \dim_k \text{Hom}(Y_i, Y) \mid Y \in \text{CM}R \text{ is indecomposable}\}$ where X_i and Y_i are in [1, Figures (1) – (7)] . For an indecomposable graded maximal Cohen-Macaulay module X , we denote by X^+ (resp. X^-) the smallest additive full subcategory of $\text{CM}R$ containing all indecomposable graded maximal Cohen-Macaulay modules Y with $X \preceq Y$ (resp. $Y \preceq X$). Then, by induction on the length of the path from X to Y (resp. from Y to X), one can easily check that $\dim_k \text{Hom}(X, Y) = 1$ (resp. $\dim_k \text{Hom}(Y, X) = 1$) for all indecomposable $Y \in X^+$ (resp. $Y \in X^-$) with Y is not free and $\tau Y \notin X^+$ (resp. $\tau^- Y \notin X^-$). Since R is a one dimensional $\mathbf{N}$ -graded Gorenstein ring of finite Cohen-Macaulay representation type, we may assume that R is one of the rings given in (1) . Thus we are able to compute $\dim_k \text{Hom}(R, R(n)) = \dim_k \text{Hom}(R(-n), R)$ for all n by Hilbert function. Since the functor $\text{Hom}(R, -)$ (resp. $\text{Hom}(-, R)$) is an exact functor on R^+ (resp. R^-), it is possible to compute $\dim_k \text{Hom}(R, Y)$ (resp. $\dim_k \text{Hom}(Y, R)$) for all $Y \in R^+$ (resp. $Y \in R^-$) by using Auslander-Reiten quiver. Since $\text{Hom}(R, Y) = 0$ (resp. $\text{Hom}(Y, R) = 0$) for all $Y \notin R^+$ (resp. $Y \notin R^-$) by lemma 2.1, it is possible to compute $\dim_k \text{Hom}(R, Y)$

and $\dim_k \operatorname{Hom}(Y, R)$ for all $Y \in \operatorname{CMR}$. For any $X \in \{X_i, Y_i\}_i$, since we have already computed $\dim_k \operatorname{Hom}(X, R(n)) = \dim_k \operatorname{Hom}(X(-n), R)$ and since $\operatorname{Hom}(X, -)$ is an exact functor on X^+ , it is also possible to compute $\dim_k \operatorname{Hom}(X, Y)$ for all $Y \in X^+$ by using Auslander-Reiten quiver. In this way we can accomplish the computation of $\dim_k \operatorname{Hom}(X, Y)$ for any indecomposable $X, Y \in \operatorname{CMR}$ and get the invariant $d_0(R)$. The result is shown in table 1. Looking at this table we have $d_0(R) \leq 9$.

TABLE 1

type	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$d_0(R)$	1	2	3	4	4	6	9

Finally, we prove (iii') $\Rightarrow$ (iii). Because we have already proved (iii') $\Rightarrow$ (i), we may assume that R is one given in (1). Since $\operatorname{Ext}_R^n(X, Y) \cong \operatorname{Ext}_R^1(\Omega^{n-1}X, Y)$ for all $n > 0$ and by lemma 2.3, it is enough to show $d_0(R) \geq d_1(R)$. For any indecomposable graded maximal Cohen-Macaulay module X , the first syzygy ΩX of X is also an indecomposable graded maximal Cohen-Macaulay module. Since there exists a natural epimorphism $\operatorname{Hom}(\Omega X, Y) \twoheadrightarrow \operatorname{Ext}_R^1(X, Y)$, one can see $d_0(R) \geq d_1(R)$ and get $d(R) \leq 9$. $\square$

Remark 3.4. Let R be a one dimensional $\mathbf{N}$ -graded Gorenstein ring of finite Cohen-Macaulay representation type with $R_0 = k$ being algebraically closed field of characteristic 0 (i.e. R is isomorphic to one of the rings given in (1)). In the above proof, we showed how to compute the invariant $d_0(R)$. Remark that we can also compute the invariant $d_n(R)$ ($n \geq 1$) by using Auslander-Reiten quiver in a similar way to this. Since $\operatorname{Ext}_R^n(X, Y) \cong \operatorname{Ext}_R^1(\Omega^{n-1}X, Y)$, we have $d_n(R) = d_1(R)$ for $n \geq 1$. We will show how to compute $d_1(R)$. For an indecomposable graded maximal Cohen-Macaulay module X , we denote by $X^{(1)}$ the smallest additive full subcategory of CMR containing all indecomposable graded non-free maximal Cohen-Macaulay modules Y with $\Omega X \preceq Y \preceq \tau X$. We also denote by $X^{(1)'}$ the smallest additive full subcategory of CMR containing all indecomposable graded non-free maximal Cohen-Macaulay modules Y with $\tau X \prec Y$ and $X \not\preceq Y$. It turns out from lemma 2.1 and lemma 2.2 that $\operatorname{Ext}_R^1(X, Y) = 0$ for all $Y \notin X^{(1)}$ and $\operatorname{Ext}_R^n(X, Y) = 0$ for all $Y \in X^{(1)'}$ and for all n . And it follows from lemma 2.3 that $\operatorname{Ext}_R^1(X, -)$ is an exact functor on $X^{(1)} \cup X^{(1)'}$. Hence it is possible to compute $d_1(R)$ (and therefore $d_n(R)$ for all $n \geq 1$) by using Auslander-Reiten quiver. The results are given in following table.

TABLE 2

	A_{2m+1}	A_{2m}	D_{2m}	D_{2m+1}	E_6	E_7	E_8
$d(R) = d_0(R)$	1	2	3	4	4	6	9
$d_n(R) \ (n \geq 1)$	1	2	1	2	3	4	6

REFERENCES

- [1] T. ARAYA: Exceptional sequences over graded Cohen-Macaulay rings, Math. J. Okayama Univ. (to appear).
- [2] E. DIETERICH: Reduction of isolated singularities, Comment. Math. Helv. **62** (1987), 654-676.
- [3] Y. YOSHINO: Brauer-Thrall type theorem for maximal Cohen-Macaulay modules, J. Math. Soc. Japan **39** (1987), 719-739.
- [4] Y. YOSHINO: Cohen-Macaulay Modules over Cohen-Macaulay Rings, London Math. Soc., Lecture Note Series vol. **146**, Cambridge U.P.(1990).

TOKUJI ARAYA

GRADUATE SCHOOL OF NATURAL SCIENCE AND TECHNOLOGY
OKAYAMA UNIVERSITY, 700-8531 JAPAN

(Received November 24, 2000)