

Mathematical Journal of Okayama University

Volume 29, Issue 1

1987

Article 18

JANUARY 1987

Some commutativity conditions

Hisao Tominaga*

*Okayama University

Copyright ©1987 by the authors. *Mathematical Journal of Okayama University* is produced by
The Berkeley Electronic Press (bepress). <http://escholarship.lib.okayama-u.ac.jp/mjou>

SOME COMMUTATIVITY CONDITIONS

HISAO TOMINAGA

Throughout, R will represent a ring with center C . Let A be a non-empty subset of R , $n > 1$ an integer, and α an endomorphism of $\{R, +, \cdot\}$. We consider the following conditions:

$(\alpha - A)_n$ $x^n - \alpha(x) \in A$ for every $x \in R$.

$(\Pi - A)_n$ If $x, y \in R$ and $x - y \in A$, then either $x^n = y^n$ or both x and y are in the centralizer of A in R .

$(* - A)_n$ For all positive integers k , every element of R can be written in the form $x^{nk} + a$ with some $x \in R$ and $a \in A$.

$P_7(n)$ $[x, y^n] = [x^n, y]$ for all $x, y \in R$.

In this brief note, we shall prove the following theorem which includes the recent result of Laffey [2].

Theorem 1. *The following statements are equivalent:*

- 1) R is commutative.
- 2) There exists a commutative non-empty subset A of R , an integer $n > 1$, and a surjective endomorphism α of $\{R, +, \cdot\}$ for which $(\alpha - A)_n$ and $(\Pi - A)_n$ are satisfied.
- 3) There exists a commutative non-empty subset A of R and an integer $n > 1$ for which $(* - A)_n$, $P_7(n)$ and $(\Pi - A)_n$ are satisfied.
- 4) There exists an integer $n > 1$ and a surjective endomorphism α of $\{R, +, \cdot\}$ for which $(\alpha - C)_n$ is satisfied.
- 5) There exists an integer $n > 1$ for which $(* - C)_n$ and $P_7(n)$ are satisfied.

Proof. Obviously, 1) implies 2) and 3). Also, one can easily see that 4) implies 5) (cf. [2, Lemmas 1 and 2]).

2) $\Rightarrow$ 4). By [3, Lemma 1 (3)], $[r^n, a] = 0$ for every $r \in R$ and $a \in A$. Hence $[\alpha(r), a] = [r^n, a] - [r^n - \alpha(r), a] = 0$. Since α is surjective, we see that a is central.

3) $\Rightarrow$ 5). Again by [3, Lemma 1 (3)], $[r^n, a] = 0$ for every $r \in R$ and $a \in A$. Let y be an arbitrary element of R . Then $y = x^n + b$ with some $x \in R$ and $b \in A$. Hence, for every $a \in A$, $[y, a] = [x^n, a] + [b, a] = 0$, and therefore a is central.

5) $\Rightarrow$ 1). By [1, Proposition 3 (ii)], there exists a positive integer h such that $[x^{nh}, y] = 0$ for all $x, y \in R$. This together with $(* - C)_n$ implies

that R is commutative.

REFERENCES

- [1] Y. HIRANO, Y. KOBAYASHI and H. TOMINAGA : Some polynomial identities and commutativity of s -unital rings, *Math. J. Okayama Univ.* 24 (1982), 7–13.
- [2] T. J. LAFFEY : On a conjecture of MacHale on the commutativity of certain rings, *Proc. Roy. Irish Acad.* 87A (1987), 63–65.
- [3] H. TOMINAGA and A. YAQUB : Some commutativity properties for rings, *Math. J. Okayama Univ.* 25 (1983), 81–86.

DEPARTMENT OF MATHEMATICS
OKAYAMA UNIVERSITY
OKAYAMA, 700 JAPAN

(Received January 24, 1987)