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ON THE KO-COHOMOLOGIES OF THE STUNTED LENS SPACES

Dedicated to Professor Hirosi Toda on his 60th birthday

AKIE TAMAMURA and SUSUMU KÔNO

1. Introduction. In this note, we consider the KO -cohomologies of the stunted lens spaces. In order to state our theorem we recall some notation in [4].

For the $(2n+1)$ -dimensional standard lens space $L^n(q) = S^{2n+1}/Z_q \bmod q$, we set

$$(1.1) \quad \begin{aligned} L_q^{2n+1} &= L^n(q), \\ L_q^{2n} &= \{[z_0, \dots, z_n] \in L^n(q) \mid z_n \text{ is real } \geq 0\}. \end{aligned}$$

We denote by $\text{ord } G$ the order of a finite group G .

The following theorem is immediately obtained (cf. [9]).

Theorem 1. *Let $q \geq 3$ be an odd integer, then*

$$(1) \quad \widetilde{KO}^{-2j-1}(L_q^{2m}/L_q^{2n}) \cong 0.$$

$$(2) \quad \text{i) } \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n}) \cong \widetilde{K}^{-2j}(L_q^{2m}/L_q^{2n})/(1-t)\widetilde{K}^{-2j}(L_q^{2m}/L_q^{2n}),$$

where $t: K \rightarrow K$ is the complex conjugation.

$$\text{ii) } \text{ord } \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n}) = q^{[(m+j)/2] - [(n+j)/2]}.$$

$$(3) \quad \widetilde{KO}^{-2j-1}(L_q^{2m}/L_q^{2n+1}) \cong \widetilde{KO}(S^{2n+2j+3}).$$

$$(4) \quad \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n+1}) \cong \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n+2}) \oplus \widetilde{KO}(S^{2n+2j+2}).$$

$$(5) \quad \widetilde{KO}^{-2j-1}(L_q^{2m+1}/L_q^{2n}) \cong \widetilde{KO}(S^{2m+2j+2}).$$

$$(6) \quad \widetilde{KO}^{-2j}(L_q^{2m+1}/L_q^{2n}) \cong \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n}) \oplus \widetilde{KO}(S^{2m+2j+1}).$$

$$(7) \quad \widetilde{KO}^{-2j-1}(L_q^{2m+1}/L_q^{2n+1}) \cong \widetilde{KO}(S^{2m+2j+2}) \oplus \widetilde{KO}(S^{2n+2j+3}).$$

$$(8) \quad \begin{aligned} \widetilde{KO}^{-2j}(L_q^{2m+1}/L_q^{2n+1}) \\ \cong \widetilde{KO}^{-2j}(L_q^{2m}/L_q^{2n+2}) \oplus \widetilde{KO}(S^{2n+2j+2}) \oplus \widetilde{KO}(S^{2m+2j+1}). \end{aligned}$$

As defined in [1], we denote by $\phi(n_1, n_2)$ the number of integers s with

$n_2 < s \leq n_1$ and $s \equiv 0, 1, 2$ or $4 \pmod{8}$, so that $\phi(n_1, n_2) = \phi(n_1, 0) - \phi(n_2, 0)$ with

$$\phi(n, 0) = \begin{cases} [n/2] & (n \equiv 0, 6 \text{ or } 7 \pmod{8}) \\ [n/2] + 1 & (\text{otherwise}). \end{cases}$$

Let $h: Z \rightarrow Z$ be the function defined by

$$(1.2) \quad h(s) = \begin{cases} 2 & (s \equiv 1 \pmod{8}) \\ 1 & (s \equiv 0 \text{ or } 2 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Main result is the following theorem.

Theorem 2. (1) If $j \equiv 0 \pmod{4}$ and $n \not\equiv 3 \pmod{4}$, then we have

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = 2^{\varphi(m+j, n-j)} q^{[m/4] - [n/4]}.$$

(2) If $j \equiv 0 \pmod{4}$ and $n \equiv 3 \pmod{4}$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \oplus Z.$$

(3) If $j \equiv 1 \pmod{4}$ and $m \geq n+2$, then the groups $\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n)$ are tabled as follows.

$m+j \pmod{4}$ $n+j \pmod{8}$	0	1	2	3
0	$Z \oplus Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$	$Z_2 \oplus Z_2$
1	$Z \oplus Z_2$	Z_2	Z_2	Z_2
2	Z	0	0	0
3	Z	0	0	0
4	Z	0	0	0
5	Z	0	0	0
6	Z	0	0	0
7	$Z \oplus Z_2$	Z_2	Z_2	Z_2

(4) If $j \equiv 2 \pmod{4}$, $n \not\equiv 1 \pmod{4}$ and $m \geq n+3$, then we have

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = 2^{h(m+j) + h(n+j)} q^{[(m+2)/4] - [(n+2)/4]}.$$

(5) If $j \equiv 2 \pmod{4}$, $n \equiv 1 \pmod{4}$ and $m \geq n+3$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \oplus Z & (n+j \equiv 3 \pmod{8}) \\ (\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1})/G) \oplus Z & (n+j \equiv 7 \pmod{8}), \end{cases}$$

where G denotes the kernel of the homomorphism

$$(p_{m,n})^!: \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n)$$

induced by the projection $p_{m,n}: L_{2q}^m/L_{2q}^n \rightarrow L_{2q}^m/L_{2q}^{n+1}$, and $\text{ord } G$ is equal to 2.

(6) If $j \equiv 3 \pmod{4}$ and $m \geq n+2$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} Z & (m \equiv 1 \pmod{4}) \\ Z_2 \oplus Z_2 & (m+j \equiv 2 \pmod{8}) \\ Z_2 & (m+j \equiv 1 \text{ or } 3 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Remark 1. If $m = n+1$, then we have a homeomorphism

$$S^j(L_{2q}^{n+1}/L_{2q}^n) \simeq S^{j+n+1},$$

and the group $\widetilde{KO}(S^{j+n+1})$ is well known.

Remark 2. If $m = n+2$, then we have homotopy equivalences

$$S^j(L_{2q}^m/L_{2q}^n) \simeq \begin{cases} S^{j-n+2} \vee S^{j+n+1} & (n: \text{ odd}) \\ S^{j-n}(L_{2q}^2) & (n: \text{ even}), \end{cases}$$

and parts (4) and (5) are true except the following cases :

i) If $j \equiv 2 \pmod{4}$ and $m+j = n+j+2 \equiv 3 \pmod{8}$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong Z_2.$$

ii) If $j \equiv 2 \pmod{4}$ and $m+j = n+j+2 \equiv 1 \pmod{8}$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong Z \oplus Z_2.$$

Remark 3. The partial results for the case $n = 0$ of Theorem 2 have been obtained in [11] (see also (2.3)).

Remark 4. The corresponding results for the case $q = 1$ of Theorem 2 are included in [6]. Our results imply

$$\widetilde{KO}^{-2j-1}(L_{2q}^m/L_{2q}^n) \cong \widetilde{KO}^{-2j-1}(RP(m)/RP(n))$$

for any q .

Consider the homomorphism

$$(p_n^m)^!: \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \rightarrow \widetilde{KO}^{-j}(L_{2q}^m)$$

induced by the projection

$$p_n^m: L_{2q}^m \rightarrow L_{2q}^m/L_{2q}^n.$$

Then, we have following result.

Corollary 3. *If $j \equiv 0 \pmod{4}$ and $n \not\equiv 3 \pmod{4}$, then the homomorphism $(p_n^m)^!$ is a monomorphism.*

We prove Theorem 2 in section 3 after recalling some known facts in section 2.

The authors would like to thank for the referee for pointing out the Additional Remark in the end of section 3.

2. Preliminaries. In this section we fix integers m and n with $0 \leq n < m$. Set $e^k = L_q^k - L_q^{k-1}$ for $0 \leq k \leq m$. Then the space L_q^m has the cell decomposition

$$\begin{aligned} L_q^m &= e^0 \cup e^1 \cup \dots \cup e^m, \\ \partial(e^{2k+1}) &= 0, \quad \partial(e^{2k}) = qe^{2k-1}. \end{aligned}$$

Thus we have

(2.1) *The cohomology groups of the pair (L_{2q}^m, L_{2q}^n) are given by*

$$\begin{aligned} H^j(L_{2q}^m, L_{2q}^n; Z) &\cong \begin{cases} Z & (j = m \equiv 1 \pmod{2} \text{ or } j = n+1 \equiv 0 \pmod{2}) \\ Z_{2q} & (j \equiv 0 \pmod{2} \text{ and } n+1 < j \leq m) \\ 0 & (\text{otherwise}), \end{cases} \\ H^j(L_{2q}^m, L_{2q}^n; Z_2) &\cong \begin{cases} Z_2 & (n < j \leq m) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

We recall some known facts which are useful for the proof of Theorem 2.

(2.2) [11, (1.2) and (1.3)].

$$\begin{aligned} Sq^2 \widetilde{H}^j(L_{2q}^m/L_{2q}^n; Z_2) &\cong \begin{cases} Z_2 & (j \equiv 2 \text{ or } 3 \pmod{4} \text{ and } n < j \leq m-2) \\ 0 & (\text{otherwise}). \end{cases} \end{aligned}$$

(2.3) [11, Theorems (0.1) and (0.2)].

(1) If $j \equiv 0 \pmod{4}$, then

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m) = 2^{\varphi(m+j,j)} q^{\lfloor m/4 \rfloor}.$$

(2) If $j \equiv 1 \pmod{4}$, then the groups $\widetilde{KO}^{-j}(L_{2q}^m)$ are tabled as follows.

$m+j \pmod{4}$ $j \pmod{8}$	0	1	2	3
1	$Z \oplus Z_2$	Z_2	Z_2	Z_2
5	Z	0	0	0

(3) If $j \equiv 2 \pmod{4}$ and $m \geq 2$, then we have

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m) = 2^{h(m+j)+h(j)} q^{\lfloor (m+2)/4 \rfloor},$$

where $h: Z \rightarrow Z$ is the function defined by (1.2).

(4) If $j \equiv 3 \pmod{4}$, then

$$\widetilde{KO}^{-j}(L_{2q}^m) \cong \begin{cases} Z & (m \equiv 1 \pmod{4}) \\ Z_2 \oplus Z_2 & (m+j \equiv 2 \pmod{8}) \\ Z_2 & (m+j \equiv 1 \text{ or } 3 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Consider the homomorphism

$$(i_n^m)^!: \widetilde{KO}^{-j}(L_{2q}^m) \rightarrow \widetilde{KO}^{-j}(L_{2q}^n)$$

induced by the inclusion

$$i_n^m: L_{2q}^n \rightarrow L_{2q}^m.$$

Then we have the following.

Lemma 2.4. (1) If $j \equiv 0 \pmod{4}$, then the homomorphism $(i_n^m)^!$ is an epimorphism.

(2) If $j \equiv 3 \pmod{4}$ and $m \geq n+3$, then the homomorphism $(i_n^m)^!$ is a zero-map.

Proof. (1) is obtained immediately from [7, Lemma 2.9]. From (4) of (2.3), (2) is clear except the case $m = n+3$ and $n+j \equiv 1 \pmod{8}$. Assume $n+j \equiv 1 \pmod{8}$, then $\widetilde{KO}^{-j}(L_{2q}^{n+2}) \cong Z_2$ and $\widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^n) \cong$

$\widetilde{KO}^{-j}(S^n(L_{2q}^2)) \cong Z_2$. In the exact sequence

$$\begin{aligned} \rightarrow \widetilde{KO}^{-j-1}(L_{2q}^{n+2}) &\xrightarrow{(i_n^{n+2})^!} \widetilde{KO}^{-j-1}(L_{2q}^n) \rightarrow \widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^n) \\ &\xrightarrow{(p_n^{n+2})^!} \widetilde{KO}^{-j}(L_{2q}^{n+2}) \xrightarrow{(i_n^{n+2})^!} \widetilde{KO}^{-j}(L_{2q}^n) \rightarrow \end{aligned}$$

associated to the cofibration

$$L_{2q}^n \xrightarrow{i_n^{n+2}} L_{2q}^{n+2} \xrightarrow{p_n^{n+2}} L_{2q}^{n+2}/L_{2q}^n,$$

the homomorphism $(i_n^{n+2})^!: \widetilde{KO}^{-j-1}(L_{2q}^{n+2}) \rightarrow \widetilde{KO}^{-j-1}(L_{2q}^n)$ is an epimorphism by (1). Hence, $(p_n^{n+2})^!$ is an isomorphism. So the homomorphism $(i_n^{n+2})^!: \widetilde{KO}^{-j}(L_{2q}^{n+2}) \rightarrow \widetilde{KO}^{-j}(L_{2q}^n)$ is a zero-map and the homomorphism $(i_n^{n+3})^!: \widetilde{KO}^{-j}(L_{2q}^{n+3}) \rightarrow \widetilde{KO}^{-j}(L_{2q}^n)$ is a zero-map. q.e.d.

Remark. By the above proof, we see that (2) of (2.4) is also true for the cases $j \equiv 3 \pmod{4}$, $m = n+2$ and $n+j \not\equiv 2 \pmod{8}$.

Now consider the Atiyah-Hirzebruch spectral sequences for $\widetilde{KO}^*(L_{2q}^m/L_{2q}^n)$:

$$\tilde{E}_2^{*,*} \cong \tilde{H}^*(L_{2q}^m/L_{2q}^n; KO^*(pt)) \Rightarrow \widetilde{KO}^*(L_{2q}^m/L_{2q}^n).$$

Then, as mentioned in [11], we have (cf. [5] and [10])

(2.5) *The differentials*

$$\begin{aligned} d_2^{i,8k}: E_2^{i,8k} &\rightarrow E_2^{i+2,8k-1}, \\ d_2^{i,8k-1}: E_2^{i,8k-1} &\rightarrow E_2^{i-2,8k-2} \end{aligned}$$

and

$$d_3^{i,8k-2}: E_3^{i,8k-2} \rightarrow E_3^{i+3,8k-4}$$

are induced by $Sq^2\rho_2$, Sq^2 and $\beta_2 Sq^2$ respectively, where ρ_2 is the reduction mod 2, Sq^2 is the Steenrod operation and β_2 is the Bockstein operation associated to the exact sequence

$$0 \rightarrow Z \rightarrow Z \rightarrow Z_2 \rightarrow 0.$$

Using (2.1), (2.2) and (2.5), we obtain the following results, where $\alpha(m, n)$ and $\beta(m, n)$ denote non-negative integers with

$$\alpha(m, n) \leq \begin{cases} 1 & (m \equiv 0, 1 \text{ or } 2 \pmod{4}) \\ 2 & (m \equiv 3 \pmod{4}), \end{cases}$$

$$\beta(m, n) \leq \begin{cases} 1 & (n \equiv 0 \text{ or } 2 \pmod{4}) \\ 2 & (n \equiv 3 \pmod{4}). \end{cases}$$

(2.6) Assume $m \geq n+3$, then we have

(1) If $j \equiv 0 \pmod{4}$, then we have

$$\begin{aligned} & \text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \\ &= \begin{cases} 2^{\varphi(m-j, n+j)-\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} q^{\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} & (m+j \equiv 0, 1, 6 \text{ or } 7 \pmod{8} \text{ and } n+j \equiv 1, 2, 4 \text{ or } 5 \pmod{8}) \\ 2^{\varphi(m+j, n+j)-\alpha(m, n)-\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} q^{\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} & (m+j \equiv 2, 3, 4 \text{ or } 5 \pmod{8} \text{ and } n+j \equiv 1, 2, 4 \text{ or } 5 \pmod{8}) \\ 2^{\varphi(m+j, n+j)-\beta(m, n)-\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} q^{\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} & (m+j \equiv 0, 1, 6 \text{ or } 7 \pmod{8} \text{ and } n+j \equiv 0 \text{ or } 6 \pmod{8}) \\ 2^{\varphi(m+j, n+j)-\alpha(m, n)-\beta(m, n)-\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} q^{\lfloor m/4 \rfloor - \lfloor n/4 \rfloor} & (m+j \equiv 2, 3, 4 \text{ or } 5 \pmod{8} \text{ and } n+j \equiv 0 \text{ or } 6 \pmod{8}). \end{cases} \end{aligned}$$

(2) If $j \equiv 1 \pmod{4}$, then we have

$$\begin{aligned} & \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \\ & \cong \begin{cases} Z & (m \equiv 3 \pmod{4} \text{ and } n+j \equiv 3, 4 \text{ or } 5 \pmod{8}) \\ 0 & (m \not\equiv 3 \pmod{4} \text{ and } n+j \equiv 2, 3, 4, 5 \text{ or } 6 \pmod{8}). \end{cases} \end{aligned}$$

and

$$\begin{aligned} & \text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \\ &= \begin{cases} 2^{2-\beta(m, n)} & (m \not\equiv 3 \pmod{4} \text{ and } n+j \equiv 0 \pmod{8}) \\ 2^{1-\beta(m, n)} & (m \not\equiv 3 \pmod{4} \text{ and } n+j \equiv 1 \text{ or } 7 \pmod{8}). \end{cases} \end{aligned}$$

(3) If $j \equiv 2 \pmod{4}$ and $n \not\equiv 1 \pmod{4}$, then we have

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = 2^{h(m+j)+h(n+j)-\lfloor (m+2)/4 \rfloor - \lfloor (n+2)/4 \rfloor} q^{\lfloor (m+2)/4 \rfloor - \lfloor (n+2)/4 \rfloor},$$

where $h: Z \rightarrow Z$ is the function defined by (1.2).

(4) If $j \equiv 3 \pmod{4}$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} Z & (m+j \equiv 0 \pmod{8} \text{ and } n \not\equiv 3 \pmod{4}) \\ 0 & (m+j \equiv 5, 6 \text{ or } 7 \pmod{8}), \end{cases}$$

and

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = \begin{cases} 2^{2-\alpha(m, n)} & (m+j \equiv 2 \pmod{8}) \\ 2^{1-\alpha(m, n)} & (m+j \equiv 1 \text{ or } 3 \pmod{4}). \end{cases}$$

3. Proof of Theorem 2. In this section we complete the proof of Theorem 2. Consider the exact sequence

$$(3.1) \quad \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \xrightarrow{(p_n^m)^!} \widetilde{KO}^{-j}(L_{2q}^m) \xrightarrow{(i_n^m)^!} \widetilde{KO}^{-j}(L_{2q}^n) \rightarrow$$

associated to the cofibration

$$L_{2q}^n \xrightarrow{i_n^m} L_{2q}^m \xrightarrow{p_n^m} L_{2q}^m/L_{2q}^n,$$

where i_n^m and p_n^m are maps given in sections 2 and 1 respectively. It follows from the exactness of (3.1) by making use of (2.3), (2.6) and Lemma 2.4 that

$$(3.2) \quad \alpha(m, n) = \beta(m, n) = 0 \text{ for } m \geq n+3.$$

(3.3) The homomorphism $(p_n^m)^!$ is an isomorphism for $j \equiv 3 \pmod{4}$ and $m \geq n+3$.

Thus, using Remarks 1 and 2, we see

(3.4) (1) If $j \equiv 0 \pmod{4}$ and $n \not\equiv 3 \pmod{4}$, then we have

$$\text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) = 2^{\varphi(m+j, n+j)} q^{\lfloor m/4 \rfloor - \lfloor n/4 \rfloor}.$$

(2) If $j \equiv 1 \pmod{4}$ and $m \geq n+2$, then we have

$$\begin{aligned} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) &\cong Z_2 \\ (m \not\equiv 3 \pmod{4} \text{ and } n+j &\equiv 1 \text{ or } 7 \pmod{8}), \end{aligned}$$

and

$$\begin{aligned} \text{ord } \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) &= 4 \\ (m \not\equiv 3 \pmod{4} \text{ and } n+j &\equiv 0 \pmod{8}). \end{aligned}$$

(3) If $j \equiv 3 \pmod{4}$ and $m \geq n+2$, then we have

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} Z & (m \equiv 1 \pmod{4}) \\ Z_2 \oplus Z_2 & (m+j \equiv 2 \pmod{8}) \\ Z_2 & (m+j \equiv 1 \text{ or } 3 \pmod{8}) \\ 0 & (\text{otherwise}). \end{cases}$$

Remark. (1) (3.4) implies Corollary 3.

(2) (3.3) is also true for the cases $j \equiv 3 \pmod{4}$, $m = n+2$ and $n+j \not\equiv 2 \pmod{8}$.

Now, consider the exact sequence

$$\widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^n) \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \rightarrow \widetilde{KO}^{-j+1}(S^{m+1})$$

associated to the cofibration

$$L_{2q}^m/L_{2q}^n \rightarrow L_{2q}^{m+1}/L_{2q}^n \rightarrow S^{m+1}.$$

Then from (3.4), we obtain

(3.5) *If $j \equiv 1 \pmod{4}$, $m \equiv 3 \pmod{4}$ and $m \geq n+2$, then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} Z \oplus Z_2 & (n+j \equiv 1 \text{ or } 7 \pmod{8}) \\ Z & (n+j \equiv 2 \text{ or } 6 \pmod{8}). \end{cases}$$

Consider the commutative diagram

$$\begin{array}{ccccc} \widetilde{KO}^{-j+1}(S^{m+1}) & = & \widetilde{KO}^{-j+1}(S^{m+1}) & & \\ \delta_1 \uparrow & & \uparrow \delta & & \\ \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+2}) & \xrightarrow{g} & \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) & \xrightarrow{f} & \widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^n) \\ \uparrow & & \uparrow k & & \parallel \\ \widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n+2}) & \xrightarrow{g_1} & \widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^n) & \xrightarrow{f_1} & \widetilde{KO}^{-j}(L_{2q}^{n+2}/L_{2q}^n) \end{array}$$

of the exact sequences, where the upper row is associated to the cofibration

$$L_{2q}^{n+2}/L_{2q}^n \rightarrow L_{2q}^m/L_{2q}^n \rightarrow L_{2q}^m/L_{2q}^{n+2}.$$

If $j \equiv 1 \pmod{4}$, $m \equiv 3 \pmod{4}$ and $n+j \equiv 0 \pmod{8}$, then $L_{2q}^{n+2}/L_{2q}^n \simeq S^{n+2} \vee S^{n+1}$, $\widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^{n+2}) \cong 0$ and $\text{ord } \widetilde{KO}^{-j}(L_{2q}^{m+1}/L_{2q}^n)$ is equal to 4. This implies that f_1 is an isomorphism and the short exact sequence

$$0 \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+2}) \xrightarrow{g} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \xrightarrow{f} \widetilde{KO}^{-j}(S^{n+2} \vee S^{n+1}) \rightarrow 0$$

splits. Thus we obtain

(3.6) *If $j \equiv 1 \pmod{4}$, $n+j \equiv 0 \pmod{8}$ and $m \geq n+2$, then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \begin{cases} Z \oplus Z_2 \oplus Z_2 & (m \equiv 3 \pmod{4}) \\ Z_2 \oplus Z_2 & (m \not\equiv 3 \pmod{4}). \end{cases}$$

Now we turn to the case $j \equiv 0 \pmod{2}$ and $n+j \equiv 3 \pmod{4}$. Consider the exact sequence

$$\begin{aligned}
 (3.7) \quad & \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^n) \xrightarrow{(i_{m,n})^!} \widetilde{KO}^{-j-1}(S^{n+1}) \\
 & \xrightarrow{\delta} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \xrightarrow{(i_{m,n})^!} \widetilde{KO}^{-j}(S^{n+1}) \\
 & \xrightarrow{\delta} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^n) \xrightarrow{(i_{m,n})^!} \widetilde{KO}^{-j+1}(S^{n+1})
 \end{aligned}$$

associated to the cofibration

$$S^{n+1} \xrightarrow{i_{m,n}} L_{2q}^m/L_{2q}^n \xrightarrow{p_{m,n}} L_{2q}^m/L_{2q}^{n+1}.$$

In this case $\widetilde{KO}^{-j+1}(S^{n+1}) \cong 0$, and

$$\mathrm{rk} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^{n+1}) = \mathrm{rk} \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^n)$$

by (2.6), (3.4), (3.5) and (3.6). Hence we have

(3.8) *If $j \equiv 0 \pmod{2}$ and $n+j \equiv 3 \pmod{4}$, then the image of the homomorphism*

$$\delta: \widetilde{KO}^{-j}(S^{n+1}) \rightarrow \widetilde{KO}^{-j+1}(L_{2q}^m/L_{2q}^{n+1})$$

has a finite order.

If $n+j \equiv 3 \pmod{8}$, then we have

$$\widetilde{KO}^{-j-1}(S^{n+1}) \cong 0.$$

It follows from (3.8) that we obtain a short exact sequence

$$0 \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \rightarrow Z \rightarrow 0.$$

Thus we have

(3.9) *If $j \equiv 0 \pmod{2}$ and $n+j \equiv 3 \pmod{8}$, then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \oplus Z.$$

Assume $j \equiv 0 \pmod{4}$, $n+j \equiv 7 \pmod{8}$, $m \equiv 3 \pmod{4}$ and $m \geq n+2$. Then we have

$$\begin{aligned}
 \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^{n+1}) &\cong Z_2, \\
 \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^n) &\cong Z_2 \oplus Z_2, \\
 \widetilde{KO}^{-j-1}(S^{n+1}) &\cong Z_2
 \end{aligned}$$

by (3.4) and (3.6). This implies that, in the exact sequence (3.7), the homomorphism

$$(i_{m,n})^!: \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^n) \rightarrow \widetilde{KO}^{-j-1}(S^{n+1})$$

is an epimorphism, and the homomorphism

$$\delta: \widetilde{KO}^{-j-1}(S^{n+1}) \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1})$$

is a zero-map. Note that the homomorphism δ is also a zero-map for the cases $m = n+1$ or $m \equiv 3 \pmod{4}$. It follows from (3.8) that we obtain a short exact sequence

$$0 \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \rightarrow Z \rightarrow 0.$$

Thus we obtain

(3.10) *If $j \equiv 0 \pmod{4}$ and $n+j \equiv 7 \pmod{8}$, then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \oplus Z.$$

Finally, assume $j \equiv 2 \pmod{4}$, $n+j \equiv 7 \pmod{8}$ and $m \geq n+3$. Then, in the exact sequence (3.7), the homomorphism

$$(p_{m,n})^!: \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^{n+1}) \rightarrow \widetilde{KO}^{-j-1}(L_{2q}^m/L_{2q}^n)$$

is an isomorphism by (3.3). Hence we obtain an exact sequence

$$0 \rightarrow Z_2 \rightarrow \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1}) \xrightarrow{(p_{m,n})^!} \widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \rightarrow Z \rightarrow 0$$

by (3.8). Thus we have

(3.11) *If $j \equiv 2 \pmod{4}$, $n+j \equiv 7 \pmod{8}$ and $m \geq n+3$, then we have*

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong (\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^{n+1})/G) \oplus Z,$$

where G denotes the kernel of the homomorphism $(p_{m,n})^!$.

Now, summarizing (2.6), (3.4), (3.5), (3.6), (3.9), (3.10) and (3.11), we obtain Theorem 2. The proof is thus completed.

Additional Remark. In view of the fact that the stunted lens spaces are Thom complexes, one may think of using the KO-theory Thom isomorphism in the computation of $\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n)$. It turns out that there is an

isomorphism

$$\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n) \cong KO^{-j-n-1}(L_{2q}^{m-n-1}) \text{ for } n \equiv 3 \pmod{4},$$

which is a generalized form of Proposition (2.1) of [6]. Combining this isomorphism with the results in Theorem 2, one can now get further information about $\widetilde{KO}^{-j}(L_{2q}^m/L_{2q}^n)$.

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