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## On Generalized PF-Rings

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## ON GENERALIZED PF-RINGS

HASAN AL-EZEH

Throughout this paper a ring denotes a commutative ring with unity. A ring  $R$  is called a PP-ring if for every  $a \in R$ , the principal ideal  $aR$  is a projective  $R$ -module. Hirano [4] defined a ring  $R$  to be a generalized PP-ring (or GPP-ring) if for every  $a \in R$ , there exists a positive integer  $n$  such that  $a^n R$  is a projective  $R$ -module. A study of this class of rings was carried by Hirano [4]. Recall that a ring  $R$  is called a PF-ring if for every  $a \in R$ , the principal ideal  $aR$  is a flat  $R$ -module. Now, we define a ring  $R$  to be a generalized PF-ring (or GPF-ring) if for every  $a \in R$ , there exists a positive integer  $n$  such that  $a^n R$  is a flat  $R$ -module. Our aim in this paper is to study the class of GPF-rings and how it is related to GPP-rings. In §1, we study some of the basic properties of GPF-rings. Then in §2, we give a different proof for a result that was proved by Hirano [4].

**1. Some results on GPF-ring.** An ideal  $I$  of a ring  $R$  is called *pure* if for every  $x \in I$ , there exists  $y \in I$  such that  $xy = x$ . It is well known that an ideal  $I$  of a ring  $R$  is pure if and only if  $R/I$  is a flat  $R$ -module, see Matlis [5]. For any  $a \in R$ , the mapping  $f: R \rightarrow aR$  defined by  $f(x) = ax$  is an  $R$ -module epimorphism. Now, the annihilator ideal,  $\text{ann}_R(a)$  is pure if and only if  $R/\text{ann}_R(a)$  is a flat  $R$ -module. Since  $R/\text{ann}_R(a)$  is isomorphic to  $aR$ ,  $aR$  is a flat  $R$ -module if and only if  $\text{ann}_R(a)$  is a pure ideal of  $R$ . Thus, we get the following easy lemma.

**Lemma 1.1.** *A ring  $R$  is a PF-ring if and only if for every  $a \in R$ ,  $\text{ann}_R(a)$  is a pure ideal of  $R$ .*

Also from the above argument we obtain the following easy lemma.

**Lemma 1.2.** *A ring  $R$  is a GPF-ring if and only if for every  $a \in R$ , there exists a positive integer  $n$  such that  $\text{ann}_R(a^n)$  is a pure ideal of  $R$ .*

Now, we prove an easy result that will be used frequently later on.

**Lemma 1.3.** *Let  $R$  be a ring, and  $a \in R$ . If  $\text{ann}_R(a)$  is pure, then*

for any positive integer  $m$ ,  $\text{ann}_R(a^m)$  is pure.

*Proof.* Let  $x \in \text{ann}_R(a^m)$ . Then  $xa^m = 0$ . If  $m = 1$ , we are done. If  $m > 1$ , then  $xaa^{m-1} = 0$ , and hence  $xa^{m-1} \in \text{ann}_R(a)$ . Since  $\text{ann}_R(a)$  is pure, there exists  $b \in \text{ann}_R(a)$  such that  $xa^{m-1}b = xa^{m-1}$ . So,  $xa^{m-1} = 0$ . Inductively, we get  $xa = 0$ . So there exists  $c \in \text{ann}_R(a)$  such that  $xc = x$ . Since  $c \in \text{ann}_R(a^m)$ , we are done.

**Corollary 1.4.** *Let  $R$  be a ring. For any  $a \in R$ , if  $aR$  is a flat  $R$ -module, then for any positive integer  $n$ ,  $a^nR$  is a flat  $R$ -module.*

For more details about PF-rings, see Matlis [5], Al-Ezeh ([1], [2]), Al-Ezeh et al. [3] and Naoum [6].

First, we characterize local GPF-rings.

**Lemma 1.5.** *A local ring  $R$  is GPF-ring if and only if every element  $a$  in  $R$  is either a non-zero-divisor or a nilpotent element.*

*Proof.* Assume that  $R$  is a local GPF-ring. Let  $a \in R$ . Since  $R$  is a GPF-ring, there exists a positive integer  $n$  such that  $\text{ann}_R(a^n)$  is pure. If  $\text{ann}_R(a^n) = 0$ , then  $a$  is a non-zero-divisor. If  $\text{ann}_R(a^n) \neq 0$ , there exists a non-zero  $b \in \text{ann}_R(a^n)$ . So there exists  $c \in \text{ann}_R(a^n)$  such that  $bc = b$ . If  $a^n \neq 0$ , then  $1 - c$  is a unit because  $R$  is local. Thus,  $b = 0$ , a contradiction. Consequently,  $a^n = 0$ , i.e.  $a$  is nilpotent.

Conversely, let  $a \in R$ . If  $a$  is a non-zero-divisor, then  $\text{ann}_R(a) = 0$  which is pure. If  $a$  is nilpotent, then there exists a positive integer  $n$  such that  $a^n = 0$ . So,  $\text{ann}_R(a^n) = R$  which is a pure ideal of  $R$ . Consequently,  $R$  is a GPF-ring.

**Lemma 1.6.** *Let  $R$  be a GPF-ring. If  $P$  is a prime ideal of  $R$ , then the localization,  $R_P$ , is a GPF-ring.*

*Proof.* Let  $a/s \in R_P$ . Since  $R$  is a GPF-ring, there exists a positive integer  $n$  such that  $a^nR$  is a flat  $R$ -module. But  $(a/s)^n = a^nR_P$ , so  $(a/s)^nR_P$  is a flat  $R_P$ -module because flatness is a local property. Thus  $R_P$  is a GPF-ring.

So, we get the following corollary which was proved differently in Matlis [5].

**Corollary 1.7.** *A ring  $R$  is a PF-ring if and only if every localization*

$R_P$  is an integral domain.

*Proof.* Assume  $R$  is a PF-ring. Taking  $n=1$  in the proof of Lemma 1.6, we get the if direction.

Conversely, let  $a \in R$ , then  $aR_P$  is a flat  $R_P$ -module since  $R_P$  is an integral domain. Because flatness is a local property,  $aR$  is a flat  $R$ -module. So  $R$  is a PF-ring.

The following theorem characterizes GPF-rings through localizations.

**Theorem 1.8.** *A ring  $R$  is a GPF-ring if and only if for every  $a \in R$  either  $a$  is a non-zero-divisor in each localization  $R_P$  or there exists a positive integer  $n$  such that  $a^n=0$  in each  $R_P$ , where  $a$  is not a zero-divisor.*

*Proof.* Assume that  $R$  is a GPF-ring. Let  $a \in R$ , then there exists a positive integer  $n$  such that  $a^n R$  is a flat  $R$ -module. So  $a^n R_P$  is a flat  $R_P$ -module, i.e.  $\text{ann}_{R_P}(a^n)$  is a pure ideal in  $R_P$ . Exactly as in the proof of Lemma 1.5, either  $a$  is a non-zero-divisor in  $R_P$  or  $a^n=0$  in  $R_P$ .

Conversely, assume that the condition holds. Let  $a \in R$ . If  $a$  is a non-zero-divisor in each  $R_P$ , then  $aR_P$  is a flat  $R_P$ -module for each  $P$ . Since flatness is a local property,  $aR$  is a flat  $R$ -module. If for some prime  $P$ ,  $a^n=0$  in  $R_P$  while for the others  $a$  is a non-zero-divisor, then  $a^n R_P$  is a flat  $R_P$ -module for all such prime ideals  $P$  of the first type. For all prime ideals of the second type,  $aR_P$  is a flat  $R_P$ -module. Consequently,  $a^n R$  is a flat  $R$ -module.

**Theorem 1.9.** *A ring  $R$  is a reduced GPF-ring if and only if  $R$  is a PF-ring.*

*Proof.* Clearly, every PF-ring is a GPF-ring. Also every PF-ring is reduced (without nontrivial nilpotent elements) see Al-Ezeh [1]. So  $R$  is a reduced GPF-ring.

Conversely, assume that  $R$  is a reduced GPF-ring. So for each prime ideal  $P$ ,  $R_P$  is a reduced GPF-ring. That is  $R_P$  is an integral domain. By Corollary 1.7,  $R$  is a PF-ring.

More generally we prove the following theorem.

**Theorem 1.10.** *Let  $R$  be a GPF-ring. Then if  $N$  is the nilradical of  $R$ ,  $R/N$  is a PF-ring.*

*Proof.* Let  $a+N \in R/N$  and  $b+N \in \text{ann}_{R/N}(a+N)$ . Then  $ba \in N$ . So, there exists a positive integer  $n$  such that  $b^n a^n = 0$ , i.e.  $b^n \in \text{ann}_R(a^n)$ . Since  $R$  is a GPF-ring, there exists a positive integer  $m$  such that  $\text{ann}_R(a^m)$  is pure. By Lemma 1.3,  $\text{ann}_R(a^{nm})$  is pure. Since  $b^n \in \text{ann}_R(a^{nm})$ , there exists  $c \in \text{ann}_R(a^{nm})$  such that  $b^n c = b^n$ . Hence  $bc - b \in N$ . Moreover,  $ca \in N$ , since  $ca^{nm} = 0$ . Thus  $c+N \in \text{ann}_{R/N}(a+N)$  and  $(b+N)(c+N) = b+N$ . Consequently,  $R/N$  is a PF-ring.

**Theorem 1.11.** *Let  $R$  be a GPF-ring. For any pure ideal  $I$  of  $R$ ,  $R/I$  is a GPF-ring.*

*Proof.* Let  $a+I \in R/I$ . Since  $R$  is a GPF-ring, there exists a positive integer  $n$  such that  $\text{ann}_R(a^n)$  is pure. Now, we want to show that  $\text{ann}_{R/I}(a^n+I)$  is pure. Let  $x+I \in \text{ann}_{R/I}(a^n+I)$ , then  $xa^n \in I$ . Since  $I$  is pure, there exists  $y \in I$  such that  $xa^n y = xa^n$ , i.e.  $a^n(xy-x) = 0$ . So, there exists  $z \in \text{ann}_R(a^n)$  such that  $(xy-x)z = xy-x$ . Thus,  $xz-x \in I$ . Hence  $(x+I)(a^n+I) = I$  and  $(x+I)(x+I) = x+I$ . Therefore,  $\text{ann}_{R/I}(a^n+I)$  is pure. Consequently,  $R/I$  is a GPF-ring.

**2. Generalized PF-rings and generalized PP-rings.** Recall that a ring  $R$  is called a  $\pi$ -regular ring if for every  $a \in R$ , there exists a positive integer  $n$  such that  $a^n = a^{2n}b$  for some  $b \in R$ , and a ring  $R$  is called quasi  $\pi$ -regular ring if the classical ring of quotient of  $R$ ,  $Q(R)$ , is a  $\pi$ -regular ring. Hirano [4] proved that  $R$  is a quasi  $\pi$ -regular ring if and only if for each  $a \in R$ , there exists a positive integer  $n$  and a non-zero-divisor  $d$  such that  $a^n d = a^{2n}$ . The following theorem was proved by Hirano [4], but we give here an alternative proof using the characterization of GPF-rings via pure ideals.

**Theorem 2.1.** *A ring  $R$  is a GPP-ring if and only if it is a quasi  $\pi$ -regular, GPF-ring.*

*Proof.* Assume  $R$  is a GPP-ring, then it is a GPF-ring. Now, let  $a \in R$ . Then there exists a positive integer  $n$  and an idempotent  $e$  such that  $\text{ann}_R(a^n) = eR$ . Then  $a^n + e$  is a non-zero-divisor and  $a^n(a^n + e) = a^{2n}$ .

Conversely, assume  $R$  is a quasi  $\pi$ -regular, GPF-ring. Let  $a \in R$ . Since  $R$  is a quasi  $\pi$ -regular ring, there exists a positive integer  $n$  and a non-zero-divisor  $d$  such that  $a^n d = a^{2n}$ . Also, since  $R$  is a GPF-ring, there exists a positive integer  $m$  such that  $\text{ann}_R(a^m)$  is pure. Now,  $a^{nm} d^m = a^{2nm}$ .

Let  $t = nm$  and  $u = d^m$ , then  $a^t u = a^{2t}$  and  $u$  is a non-zero-divisor. By Lemma 1.3,  $\text{ann}_R(a^t)$  is pure. If  $b = u = a^t$ , then  $b \in \text{ann}_R(a^t)$ . Since  $\text{ann}_R(a^t)$  is pure, there exists  $e \in \text{ann}_R(a^t)$  such that  $be = b$ . Now, consider  $ue(1-e) = (u-a^t)e(1-e) = be(1-e) = 0$ . Thus  $e(1-e) = 0$ . So  $e$  is an idempotent element. Clearly,  $eR \subset \text{ann}_R(a^t)$ . Now, let  $x \in \text{ann}_R(a^t)$ . Then  $xa^t = 0$ .

Consider

$$x(1-e)u = x(1-e)(u-a^t) = x(1-e)b = 0.$$

Thus  $x(1-e) = 0$ , i.e.  $x = xe$ . Therefore  $\text{ann}_R(a^t) = xe$ . Hence  $R$  is a GPP-ring.

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