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SOME DECOMPOSITION THEOREMS FOR RINGS

Dedicated to Professor Miyuki Yamada on his 60th birthday

HIROAKI KOMATSU and HISAO TOMINAGA

Throughout, R will represent a ring with center C . An element x of R is called a *right p.p. element* if there exists an idempotent e in R such that $xe = x$ and $|r \in R \mid xr = 0| = |r \in R \mid er = 0|$ (see [5]). Let P be the set of right p.p. elements in R , and N the set of nilpotent elements in R . Following [3], R is called a P_k -ring if $xR^k = xR^k x$ for all $x \in R$; following [7], R is called a J_{k+1} -ring if for each $x_1, \dots, x_{k+1} \in R$ there exists an integer $n > 1$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^n$, where k is a fixed positive integer.

The main theorems of the present paper are stated as follows :

Theorem 1. (1) *The following are equivalent :*

1) *For each sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements in R there exist positive integers k, k' such that $x_1 \cdots x_{k+1} = x_1^{k+2} z$ and $x_{k'+1} \cdots x_1 = x_1^{k'+2} z'$ with some $z, z' \in R$.*

2) *For each sequence $\{x_i\}_{i \in \mathbb{N}}$ of elements in R there exist positive integers k, k' such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$ and $x_{k'+1} \cdots x_1 = (x_{k'+1} \cdots x_1)^2 z'$ with some $z, z' \in R$.*

3) *$R = N \oplus P$ (strictly speaking, both N and P are ideals of R and R is the direct sum of N and P), P is strongly regular, and N is (right and left) T -nilpotent.*

(2) *Let k be a positive integer. Then the following are equivalent :*

1) *R is a P_k -ring.*

2) *For each $x_1, \dots, x_{k+1} \in R$ there exists $z \in R$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$.*

3) *$R = N \oplus P$, P is strongly regular, and $N^{k+1} = 0$.*

(3) *Let k be a positive integer. Then the following are equivalent:*

1) *R is a J_{k+1} -ring.*

2) *For each $x_1, \dots, x_{k+1} \in R$ there exists $z \in \langle x_1, \dots, x_{k+1} \rangle$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$.*

3) *$R = N \oplus P$, P is a J -ring, and $N^{k+1} = 0$.*

Theorem 2. *Let k be a positive integer, and A an additive subsemi-group of R with $A \subseteq N$. Suppose that for each $x_1, \dots, x_{k+1} \in R \setminus A$ there exists $z \in R$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$. Then $R = N \oplus P$, P is strongly regular, and $N^{k+1} = 0$.*

Proof of Theorem 1. (1) We can easily see that 3) implies 1) and 2).

1) $\Rightarrow$ 3). Let $\{x_i\}_{i \in \mathbb{N}}$ be an arbitrary sequence of elements in R with $x_1 \in N$. Then there exists $k_1 > 0$ such that $x_1 \cdots x_{k_1+1} = x_1^{k_1+2} z_1$ with some $z_1 \in R$. Next, there exists $k_2 > 0$ such that $x_1^{k_1+2} z_1 x_{k_1+2} \cdots x_{k_1+k_2} = x_1^{(k_1+2)(k_2+2)} z_2$ with some $z_2 \in R$, and so $x_1 \cdots x_{k_1+k_2} = x_1^{(k_1+2)(k_2+2)} z_2$. Continuing the same procedure, we obtain eventually a positive integer k such that $x_1 \cdots x_{k+1} = 0$; similarly, we can find a positive integer k' such that $x_{k'+1} \cdots x_1 = 0$. In particular, for any idempotent e of R we have $Ne = eN = 0$. Hence e is central, and so all the idempotents of R generate a reduced ideal. Furthermore, for each $x \in R$ there exists $h > 0$ such that $x^{h+1} \in x^{h+2}R$, and so R is right π -regular. Hence, by a result of Zöschinger-Dischinger (see, e.g., [4, Proposition 2]), R is strongly π -regular. Now, by [5, Corollary 1], $R = N \oplus P$, P is strongly regular, and N is T -nilpotent.

2) $\Rightarrow$ 3). We claim first that every idempotent e of R belongs to C . Actually, for any $x \in R$ there exist $z, z' \in R$ such that $(x - ex)e = ((x - ex)e)^2 z = 0$ and $e(x - xe) = (e(x - xe))^2 z' = 0$, and therefore $xe = exe = ex$.

For each $x \in R$ there exists a positive integer k such that $x^{k+1} \in x^{2(k+1)}R$. Hence, again by a result of Zöschinger-Dischinger, R is strongly π -regular. Now, let e be an arbitrary (central) idempotent of R , and $a \in N$: $a^n = 0$. Then $ea \in (ea)^2R = (ea)^nR = a^n eR = 0$, and therefore $eR \cap N = 0$. Hence, by [5, Corollary 1], $R = N \oplus P$ and P is strongly regular. If $\{x_i\}_{i \in \mathbb{N}}$ is an arbitrary sequence of elements in R such that x_i is in the nil ideal N , then we can easily see that there exist positive integers k, k' such that $x_1 \cdots x_{k+1} = x_{k'+1} \cdots x_1 = 0$.

(2) The equivalence of 1) and 3) has been proved in [5, Corollary 2]. Careful scrutiny of the proof of (1) shows that 2) and 3) are equivalent.

(3) Obviously, 3) $\Rightarrow$ 1) $\Rightarrow$ 2).

2) $\Rightarrow$ 3). By (2), $R = N \oplus P$, P is strongly regular, and $N^{k+1} = 0$. For any $x \in R$ there exists $z \in \langle x \rangle$ such that $x^{k+1} = x^{2(k+1)}z$. Hence R is

periodic by Chacron's criterion [2], and so P is a J -ring. (This implication is also an easy consequence of [6, Theorem 1].)

Theorem 1 (3) improves [1, Theorem 1] as well as [7, Theorem 2.4], and enables us to generalize [1, Theorem 2] as follows :

Corollary 1. *Let k be a positive integer. If for each $x_1, \dots, x_{k+1} \in R$ there exists $z \in \langle x_1, \dots, x_{k+1} \rangle$ such that $x_1 \cdots x_{k+1} = (x_{k+1} \cdots x_1)^2 z$, then $R = N \oplus P$, P is a J -ring, and $N^{k+1} = 0$.*

Proof. There exists $z' \in \langle x_1, \dots, x_{k+1} \rangle$ such that $x_{k+1} \cdots x_1 = (x_1 \cdots x_{k+1})^2 z'$. Hence $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z'(x_{k+1} \cdots x_1)z$, and the assertion is clear by Theorem 1 (3).

Proof of Theorem 2. As is easily seen,

(*) if $x_1, \dots, x_{k+1} \in R \setminus A$ and $x_1 \cdots x_{k+1} \in N$ then $x_1 \cdots x_{k+1} = 0$; in particular, if $b \in N \setminus A$ then $b^{k+1} = 0$.

Next, we claim that

(**) if $b \in N \setminus A$ and $x \in R$ then $bx \in N$ and $xb \in N$.

In order to see this, we may assume $bx \notin A$. Noting that $b^{k+1} = 0$, we see that $b^{k-1}bxb \in N$, and so $b^{k-1}bxb = 0$ by (*). Hence $b^{k-1}(bx)^2 = 0$. Next, by making use of the fact that $b^{k-2}(bx)^2b \in N$ we can see that $b^{k-2}(bx)^3 = 0$. Continuing the same procedure, we obtain eventually $(bx)^{k+1} = 0$. Hence $bx \in N$; similarly, $xb \in N$.

Now, let $x_1, \dots, x_{k+1} \in R \setminus A$ and $a_1, \dots, a_{k+1} \in A$. If x_i is in N then $x_i \cdots x_{k+1} x_1 \cdots x_{i-1} = 0$ by (**) and (*), and therefore $x_1 \cdots x_{k+1} = 0$, again by (**) and (*); namely $(R \setminus A)^{i-1}(N \setminus A)(R \setminus A)^{k+1-i} = 0$. Noting that $x_j - a_j \in R \setminus A$ ($j \neq i$), we get $x_1 \cdots a_j \cdots x_{k+1} = x_1 \cdots x_{k+1} - x_1 \cdots (x_j - a_j) \cdots x_{k+1} = 0$, and repeating this procedure, we can easily see that $R^{i-1}(N \setminus A)R^{k+1-i} = 0$. Furthermore, if $a_i^s = 0$ then $(x_i - a_i)^{(k+1)s} = (-a_i)^{(k+1)s} = 0$, and so $x_i - a_i \in N \setminus A$. Hence we can see that $R^{i-1}AR^{k+1-i} = 0$ ($i = 1, \dots, k+1$). Now the assertion is clear by Theorem 1 (2).

Corollary 2. *Let k be a positive integer, and A an additive subsemi-group of R properly contained in the Jacobson radical J of R . Suppose that for each $x_1, \dots, x_{k+1} \in R \setminus A$ there exists $z \in R$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$. Then $R = N \oplus P$, P is strongly regular, and $N^{k+1} = 0$.*

Proof. In view of Theorem 2, it suffices to show that $J^{k+1} = 0$. Let $x_1, \dots, x_{k+1} \in J \setminus A$ and $a_1, \dots, a_{k+1} \in A$. Then there exists $z \in R$ such

that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z \in x_1 \cdots x_{k+1} J$. Hence $x_1 \cdots x_{k+1} = 0$; namely $(J \setminus A)^{k+1} = 0$. Since $x_i - a_i \in J \setminus A$ for all i , we can easily see that $J^{k+1} = 0$ (see the proof of Theorem 2).

Finally, the following is immediate by Theorem 2, Corollary 2 and the proof of Theorem 1 (3), and generalizes [1, Theorems 3 and 4].

Corollary 3. *Let k be a positive integer, and A an additive subsemi-group of R with $A \subseteq N$ (or $A \subseteq J$). Suppose that for each $x_1, \dots, x_{k+1} \in R \setminus A$ there exists $z \in \langle x_1, \dots, x_{k+1} \rangle$ such that $x_1 \cdots x_{k+1} = (x_1 \cdots x_{k+1})^2 z$. Then $R = N \oplus P$, P is a J -ring, and $N^{k+1} = 0$.*

REFERENCES

- [1] H. E. BELL and S. LIGH : Some decomposition theorems for periodic rings and near rings, Math. J. Okayama Univ. 31 (1989), 93–99.
- [2] M. CHACRON : On a theorem of Herstein, Canad. J. Math. 21 (1969), 1348–1353.
- [3] K. CHIBA and H. TOMINAGA : On generalizations of P_1 -rings and gsr -rings, Math. J. Okayama Univ. 18 (1976), 149–152.
- [4] Y. HIRANO : Some studies on strongly π -regular rings, Math. J. Okayama Univ. 20 (1978), 141–149.
- [5] Y. HIRANO and H. TOMINAGA : Rings decomposed into direct sums of nil rings and certain reduced rings, Math. J. Okayama Univ. 27 (1985), 35–38.
- [6] Y. HIRANO, H. TOMINAGA and A. YAQUB : On rings in which every element is uniquely expressible as a sum of a nilpotent element and a certain potent element, Math. J. Okayama Univ. 30 (1988), 33–40.
- [7] M. Ó SEARCÓID : A structure theorem for generalized J -rings, Proc. Roy. Irish Acad. 87A (1987), 117–120.

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