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ON GALOIS H -DIMODULE ALGEBRAS

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Let k be a commutative ring and H a finite commutative, cocommutative Hopf k -algebra with dual $H^* = \text{Hom}_k(H, k)$. An H -Galois extension A of k is called a Galois H -dimodule algebra, [1], [4], [5], if A has an additional structure of an H -module (algebra) satisfying

$$h(ga) = g(ha), \quad a \in A, h \in H, g \in H^*.$$

By definition, a morphism of Galois H -dimodule algebras is an H -linear morphism of the underlying Galois extensions. It is shown in this paper that the set $D(H)$ of isomorphism classes of Galois H -dimodule algebras decomposes into

$$(1) \quad D(H) = E(H) \times \text{Hopf}_k(H, H^*)$$

where $E(H)$ is the group of isomorphism classes of H -Galois extensions of k , and $\text{Hopf}_k(H, H^*)$ the group of Hopf algebra homomorphisms $H \rightarrow H^*$. In [1] a certain product for Galois H -dimodule algebras was defined, and [1], prop. 2.4, says that $D(H)$ is a group. This generalizes [5], theorem 1.9, which states that $D(H)$ is a monoid if H is a free k -module. Under (1) the product reads

$$(2) \quad (A, \varphi) \cdot (B, \psi) = (A^\varphi \cdot B, \varphi\psi)$$

where A^φ equals A as H -comodule and has multiplication

$$a \cdot b = \sum a_{(0)}(\varphi(a_{(1)})b), \quad a, b \in A.$$

But if the product (2) is associative, or has a unit element, then $A^\varphi \cong A$ for all A and φ , and we shall give examples where $H^\varphi \not\cong H$. Consequently, we obtain that the above mentioned results are not correct, even in the special cases considered in [5], theorems 2.4 and 2.6.

Notations. The counit of H (or H^*) will be denoted by ϵ , its antipode by λ . For H -Galois extensions A and B of k their product $A \cdot B$ is defined to be the equalizer of the comodule structures

$$A \cdot B \rightarrow A \otimes B \rightrightarrows A \otimes B \otimes H,$$

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where $\otimes = \otimes_k$. We refer to [2] for a proof that $A.B$ agrees with various other definitions of the product of Galois extensions.

In the following let A be an H -Galois extension of k . Thus A is an H -comodule algebra and the map

$$\omega: H^* \otimes A \rightarrow \text{End}_k(A), \quad \omega(g \otimes b)(a) = (ga)b,$$

is an isomorphism where $ga = \sum a_{(0)} \langle g, a_{(1)} \rangle$, $g \in H^*$, $a, b \in A$, (see for example [8]). Also, ω induces an isomorphism

$$(3) \quad H^* \xrightarrow{\sim} \text{End}_{H^*}(A).$$

Indeed, consider $w \in H^* \otimes A$ such that $\omega(w)$ is H^* -linear. Then since $g(ab) = \sum g_{(1)}a \cdot g_{(2)}b$ for all $g \in H^*$, and $a, b \in A$, we have

$$w(g \otimes 1) = (\sum g_{(1)} \otimes g_{(2)})w, \quad g \in H^*.$$

Therefore, $\varepsilon(g)w = (\sum \lambda(g_{(1)})g_{(2)} \otimes 1)w = \sum (\lambda(g_{(1)}) \otimes 1)w(g_{(2)} \otimes 1) = \sum (\lambda(g_{(1)})g_{(2)} \otimes g_{(3)})w = (1 \otimes g)w$. Hence $w \in H^* \otimes A^{H^*} \cong H^*$. Since A is finitely generated and projective over H^* , (3) says that A is an invertible H^* -module.

Assume in the following that A is also an H -module satisfying $h(ga) = g(ha)$ for all $h \in H$, $g \in H^*$, and $a \in A$. By (3) this means we are given a k -algebra homomorphism $\varphi: H \rightarrow H^*$ such that

$$(4) \quad ha = \varphi(h)a = \sum a_{(0)} \langle \varphi(h), a_{(1)} \rangle, \quad a \in A, h \in H.$$

Proposition 1. *A is an H -module algebra with respect to (4) if and only if φ is a homomorphism of Hopf k -algebras.*

Proof. That A be an H -module algebra means

$$(5) \quad \varphi(h)(ab) = \sum \varphi(h_{(1)})a \cdot \varphi(h_{(2)})b, \quad h \in H, a, b \in A,$$

and $\varepsilon(h)1_A = \varphi(h)1_A = \varepsilon\varphi(h)1_A$. Since $\varphi(h)(ab) = \sum \varphi(h)_{(1)}a \cdot \varphi(h)_{(2)}b$, A is an H -module algebra if φ is a coalgebra morphism. Conversely, let $x, y \in H$ and choose $\sum_i a_i \otimes b_i$ and $\sum_j c_j \otimes d_j$ in $A \otimes A$ such that

$$(6) \quad 1 \otimes x = \sum a_i b_{i(0)} \otimes b_{i(1)}$$

and $1 \otimes y = \sum c_{j(0)}d_j \otimes c_{j(1)}$ holds in $A \otimes H$. Then assuming (5) we have

$$\begin{aligned}
 \langle \varphi(h), xy \rangle &= \sum a_i b_{i(0)} c_{j(0)} d_j \langle \varphi(h), b_{i(1)} c_{j(1)} \rangle \\
 &= \sum a_i (\varphi(h)(b_i c_j)) d_j \\
 &= \sum a_i (\varphi(h_{(1)}) b_i) (\varphi(h_{(2)}) c_j) d_j \\
 &= \sum \langle \varphi(h_{(1)}), x \rangle \langle \varphi(h_{(2)}), y \rangle.
 \end{aligned}$$

Since $\langle \varphi(h), xy \rangle = \sum \langle \varphi(h)_{(1)}, x \rangle \langle \varphi(h)_{(2)}, y \rangle$ this implies that φ is a coalgebra morphism as required.

Given $\varphi \in \text{Hopf}_k(H, H^*)$ we define on A a new multiplication by

$$a \cdot b = \sum a_{(0)} b_{(0)} \langle \varphi(a_{(1)}), b_{(1)} \rangle, \quad a, b \in A.$$

This gives us an H -comodule algebra A^φ having the same comodule structure as A .

Lemma 2. A^φ is an H -Galois extension of k .

Proof. Let $u \in H^* \otimes H^*$ denote the preimage of φ under the canonical isomorphism $H^* \otimes H^* \xrightarrow{\sim} \text{Hom}_k(H, H^*)$, $(g \otimes g')(h) = \langle g, h \rangle g'$. Then u is a 2-cocycle, (cf. [6], sec. 1.5), and if $A = H$ then A^φ coincides with the (associative) H -Galois algebra $(H, \bar{u})$ of [3], prop. 1.6. But

$$A^\varphi \xrightarrow{\sim} A \cdot H^\varphi \subset A \otimes H, \quad a \rightarrow \sum a_{(0)} \otimes a_{(1)},$$

is an isomorphism of H -comodule algebras, and this completes the proof.

Obviously $A^\epsilon = A$ and since the product in $\text{Hopf}_k(H, H^*)$ is defined by $(\varphi\psi)(h) = \sum \varphi(h_{(1)}) \psi(h_{(2)})$, $h \in H$, one has

$$(7) \quad (A^\varphi)^\psi = A^{\varphi\psi}.$$

Let $D(H)$ denote the set of isomorphism classes of Galois H -dimodule algebras.

Theorem 3. By viewing each Galois H -dimodule algebra as a pair (A, φ) as above one obtains a bijection

$$(8) \quad D(H) \xrightarrow{\sim} E(H) \times \text{Hopf}_k(H, H^*).$$

It transforms the product of Galois H -dimodule algebras of [1] into

$$(9) \quad (A, \varphi) \cdot (B, \psi) = (A^\varphi \cdot B, \varphi\psi)$$

where $A^\varphi \cdot B$ denotes the product of the Galois extensions A^φ and B .

Proof. Let $f: (A, \varphi) \rightarrow (B, \psi)$ be a homomorphism of Galois H -dimodule algebras; to prove (8) it suffices to show $\varphi = \psi$. Let $x \in H$ and write $1 \otimes x$ in $A \otimes H$ as in (6). So $1 \otimes x = \sum f(a_i)f(b_{i(0)}) \otimes b_{i(1)}$. Since $f(\varphi(h)b_i) = \psi(h)f(b_i)$ for $h \in H$ we obtain

$$\begin{aligned}\langle \varphi(h), x \rangle &= \sum f(a_i)f(b_{i(0)}) \langle \varphi(h), b_{i(1)} \rangle \\ &= \sum f(a_i)f(b_{i(0)}) \langle \psi(h), b_{i(1)} \rangle = \langle \psi(h), x \rangle.\end{aligned}$$

Thus $\varphi(h) = \psi(h)$ and we are done. Next consider the product $A.B$ of any Galois H -dimodule algebras $A = (A, \varphi)$ and $B = (B, \psi)$. By definition, [1], 2.1, we have $A.B = (A^\varphi.B, \varphi\psi)$ as H -comodules. Furthermore, also the H -module structures are the same. For let $h \in H$ and $\sum_i a_i \otimes b_i \in$

$A.B$; by abuse of notation we denote the latter by $a \otimes b$. Then

$$\begin{aligned}h.(a \otimes b) &= \sum h_{(1)}.a \otimes h_{(2)}.b \\ &= \sum a_{(0)} \langle \varphi(h_{(1)}), a_{(1)} \rangle \otimes b_{(0)} \langle \psi(h_{(2)}), b_{(1)} \rangle \\ &= \sum a_{(0)} \langle \varphi(h_{(1)}), a_{(1)} \rangle \otimes \langle \psi(h_{(2)}), a_{(2)} \rangle b \\ &= \sum a_{(0)} \langle \varphi(h_{(1)})\psi(h_{(2)}), a_{(1)} \rangle \otimes b \\ &= (\varphi\psi)(h)(a \otimes b).\end{aligned}$$

Finally, multiplication in $A.B$ is defined by the smash product formula. This means

$$\begin{aligned}(a \otimes b).(c \otimes d) &= \sum a(b_{(1)}c) \otimes b_{(0)}d \\ &= \sum ac_{(0)} \langle \varphi(b_{(1)}), c_{(1)} \rangle \otimes b_{(0)}d \\ &= \sum a_{(0)}c_{(0)} \langle \varphi(a_{(1)}), c_{(1)} \rangle \otimes bd\end{aligned}$$

since $\sum a \otimes b_{(0)} \otimes b_{(1)} = \sum a_{(0)} \otimes b \otimes a_{(1)}$. Hence $A.B = A^\varphi.B$ as k -algebras and this completes the proof.

Lemma 4. *If the product (9) is associative, or has a unit element, then $A^\varphi \cong A$ (as Galois extensions) for all $\varphi \in \text{Hopf}_k(H, H^*)$ and H -Galois extensions A of k .*

Proof. Associativity of (9) implies therefore $(A^\varphi B)^\varphi \cong A^\varphi B^\varphi$ because $E(H)$ is a group. Since $(\lambda\varphi).\varphi = \varepsilon$ holds in $\text{Hopf}_k(H, H^*)$, we have $(H^{\lambda\varphi})^\varphi = H^\varepsilon = H$. Setting $A = H^{\lambda\varphi}$ we obtain $B^\varphi \cong B^\varphi$ by (7). Hence $B^\varphi \cong B$. The other statement is evident.

Remark 5. Given H -dimodule algebras A, B , and C , the natural map

$A \# (B \# C) \rightarrow (A \# B) \# C$ is an isomorphism of H -dimodule algebras by [4], thm. 3.3. But in general the inclusion $B.C \rightarrow B \# C$ is not H -colinear, because the comodule structure of $B \# C$ is given by $b \otimes c \rightarrow \sum b_{(0)} \otimes c_{(0)} \otimes b_{(1)}c_{(1)}$, whereas that of $B.C$ by $b \otimes c \rightarrow \sum b_{(0)} \otimes c \otimes b_{(1)} = \sum b \otimes c_{(0)} \otimes c_{(1)}$. It follows that, in general, $A.(B.C)$ is not a subalgebra of $A \# (B \# C)$.

In the following, let $\bar{A}$ for $A = (A, \varphi)$ be defined as in [1], and let A^{-1} be the H -Galois extension A^{op} having H -comodule structure $a \rightarrow \sum a_{(0)} \otimes \lambda(a_{(1)})$. We claim that $\bar{A} = ((A^{-1})^{\varphi}, \lambda\varphi)$. It is clear that both sides have the same H -comodule structure, and that they are equal to A as H -modules. Furthermore, by [1], prop. 1.2c), multiplication of $\bar{A}$ is given by $a.b = \sum b_{(0)}a_{(0)}\langle \varphi(a_{(1)}), b_{(1)} \rangle$ since $\lambda^2 = \text{id}$. However, $\langle \varphi(\alpha), \beta \rangle = \langle \varphi(\alpha), \lambda^2(\beta) \rangle = \langle \lambda\varphi(\alpha), \lambda(\beta) \rangle = \langle \varphi(\lambda(\alpha)), \lambda(\beta) \rangle$ for $\alpha, \beta \in H$. Hence the multiplication of $\bar{A}$ coincides with that of $(A^{-1})^{\varphi}$, and this proves our claim. Obviously, one has $A.\bar{A} = A.A^{-1}$ as H -comodules but, in general, not as k -algebras. Consider then the isomorphism

$$(10) \quad A \otimes A \xrightarrow{\sim} A \otimes H, \quad a \otimes b \rightarrow \sum ab_{(0)} \otimes \lambda(b_{(1)}).$$

It is shown in the proof of [1], prop. 2.4, that restricting (10) gives an isomorphism of H -comodules

$$f: A.\bar{A} \xrightarrow{\sim} k \otimes H = H.$$

Furthermore, it is claimed there that f is also a morphism of k -algebras. But in general this fails to be true. Actually, f respects multiplication of $A.A^{-1}$. Indeed, let (by abuse of notation) $a \otimes b, c \otimes d \in A.A^{-1}$ and write $f(c \otimes d) = 1 \otimes h$. Then

$$\begin{aligned} f(ac \otimes db) &= \sum acd_{(0)}b_{(0)} \otimes \lambda(b_{(1)})\lambda(d_{(1)}) \\ &= \sum (a \otimes \lambda(b_{(1)}))(cd_{(0)} \otimes \lambda(d_{(1)}))(b_{(0)} \otimes 1) \\ &= \sum (a \otimes \lambda(b_{(1)}))(b_{(0)} \otimes h) = f(a \otimes b)f(c \otimes d). \end{aligned}$$

In other words, $f((a \otimes b^{\text{op}})(c \otimes d^{\text{op}})) = f(a \otimes b^{\text{op}})f(c \otimes d^{\text{op}})$, so that $f: A.A^{-1} \rightarrow H$ is an algebra homomorphism. This shows that A^{-1} represents the inverse of the class of A in the abelian group $E(H)$. (Actually the H -Galois extensions of k form a symmetric Picard category in the sense of [9]).

Remark 6. The Hopf k -algebra homomorphism $\varphi: H \rightarrow H^*$ is bijective if and only if (A, φ) is an H^* -Galois extension of k . For example this is

the case for Kummer extensions of k , cf. [7].

Let $H = k[G]$, G a finite abelian group. Then one has a natural isomorphism of $\text{Hopf}_k(H, H^*)$ with the group of bimultiplicative mappings of $G \times G$ into the group k^* of units of k . Then $k[G]^\varphi \cong k[G]$ if and only if $\varphi: G \times G \rightarrow k^*$ is a 2-coboundary, and in general there are pairings φ which define non-trivial cocycles, even if G has order two, cf. [10], p. 160, remark.

Finally let k be a field of characteristic 2, and let $H = k[X]/X^2 = k[\delta]$ be as in [5], 2.2. Let $(1, \delta^*)$ be the dual basis of $(1, \delta)$. Then

$$\eta: k^+ \rightarrow \text{Hopf}_k(H, H^*), \quad \eta(t)(\delta) = t\delta^*, \quad t \in k,$$

is an isomorphism of groups. Further, one has $H^{\eta(t)} \cong k[X]/(X^2 - t)$. So $H \cong H^{\eta(t)}$ implies that t is a square in k .

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