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Some Generalizations of Boolean Rings

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SOME GENERALIZATIONS OF BOOLEAN RINGS

HIROAKI OKAMOTO, JULIE GROSEN and HIROAKI KOMATSU

Throughout, R will represent a ring with center C . Let N denote the set of nilpotents in R , and N^* the subset of N consisting of all elements in R which square to zero. Let E be the set of idempotents in R . If $E \subseteq C$ then R is called *normal*. In case R has 1, we denote by U the multiplicative group of units of R . Following [10], R is called *(E-N) representable*, if each $x \in R$ can be written uniquely in the form $x = e + a$, where $e \in E$ and $a \in N$. Given $x \in R$, we define inductively $x^{(1)} = x$, $x^{(k)} = x^{(k-1)} \circ x$, where $x \circ y = x + y + xy$. In [8], Hirano, Komatsu, Tominaga and Yaqub considered the following condition which arose, presumably, in connection with logic: (*) for any $x, y \in R$, $(x + xy) \circ (y + yx) = 0$ if and only if $x = y$, and proved that R satisfies (*) if and only if R is commutative, R/N is a Boolean ring and $a^{(2)} = 0$ for all $a \in N$ (see Theorem 1 below). Obviously, every Boolean ring satisfies the condition (*). If R has 1, then (*) becomes (*)' for any $x, y \in R$, $(1 + x + xy)(1 + y + yx) = 1$ if and only if $x = y$. Recently, Groesen [4] gave a number of characterizations of a ring with 1 in which the condition (*)' holds.

An element x in R is called *strongly regular*, if there exist $y, y' \in R$ such that $x^2y = x = y'x^2$. As is well-known, if x is strongly regular, there exists (uniquely) $z \in R$ such that $x^2z = x$, $z^2x = z$ and $xz = zx$; furthermore, z commutes with every element which commutes with x . We denote by S the set of strongly regular elements in R . A ring R is called a *B'-ring* if $S = E$. Obviously, every Boolean ring is a *B'-ring*.

A ring R is called *s-unital* if $x \in Rx \cap xR$ for all $x \in R$, or equivalently if for each finite subset F of R there exists $e \in R$ such that $ex = x = xe$ for all $x \in F$ (see [6]). Following [11], R is called an *s*-unital ring* if for each $x \in R$ there exist $e', e'' \in E$ such that $xe' = x = e''x$, or equivalently if for each finite subset F of R there exists $e \in E$ such that $eFe = F$ (see [11, Corollary 7]). As is easily seen, every *s-unital* π -regular ring is *s*-unital*. In what follows, we shall use freely this fact. A ring R is a *cs*-unital ring* if for each $x \in R$ there exists a central idempotent e such that $ex = x$.

A ring R is called an *I-ring* (resp. *N-ring*) if every element of R is expressible as a product of elements in E (resp. N); R is called an *NI-ring* (or *I'-ring*) if every element of R is expressible as a product of elements in

$E \cup N$ (see [1] and [7]). Needless to say, every Boolean ring is an I -ring.

Our present objective is to improve several results of Groesen obtained in [4, § 5] and the main theorems of Abu-Khuzam [1] and reprove the main theorems of [2].

First, as preliminaries, we state the following lemmas.

Lemma 1 ([10, Theorem 4]). *The following are equivalent :*

- 1) R is $(E-N)$ representable.
- 2) R is normal, and every element of R can be written as a sum of an idempotent and a nilpotent element.
- 3) R is normal and $x - x^2 \in N$ for every $x \in R$.
- 4) R is normal, N is an ideal and R/N is a Boolean ring.

Lemma 2 ([8, Lemma 5]). *Let $f(X) = k_1X + k_2X^2 + \cdots + k_mX^m$ be a polynomial in $XZ[X]$ with $(k_1, k_2) = 1$. If N satisfies the identity $f(X) = 0$, then N satisfies the identities $X^3 = 0 = k_1X + (k_2 - k_1)X^2$.*

Lemma 3. *If N is closed under $\circ$ (in particular, if N is an ideal) and satisfies the identity $X^{(2)} = 0$, then N is commutative.*

Proof. For any $a, b \in N$, $a \circ b = a \circ (a \circ b)^{(2)} \circ b = a^{(2)} \circ (b \circ a) \circ b^{(2)} = b \circ a$, whence $ab = ba$ follows.

Lemma 4. (1) *If R satisfies the identity $(X + X^2)^{(2)} = 0$, then $8x = 0$, $x^5 = x^3$ and $x - x^2 \in N$ (or $x + x^2 \in N$) for all $x \in R$, and $a^3 = 0 = a^{(2)}$ for all $a \in N$.*

(2) *If N satisfies the identity $(X + X^2)^{(2)} = 0$, then $4a = 0$ and $a^3 = 0 = a^{(2)}$ for all $a \in N$.*

Proof. (1) Since $6x^2 + 2x^4 = (x + x^2)^{(2)} + (-x + (-x)^2)^{(2)} = 0$ and $4x + 4x^3 = (x + x^2)^{(2)} - (-x + (-x)^2)^{(2)} = 0$, we get $8x = (4x + 4x^3)(2 + x^2) - 2(6x^2 + 2x^4)x = 0$. Further, noting that $2x + 3x^2 + 2x^3 + x^4 = (x + x^2)^{(2)} = 0$, we can easily see that $a^3 = 0 = a^{(2)}$ for all $a \in N$ (Lemma 2). Since $(x + x^2)^6 = |(x + x^2)^{(2)} - 2(x + x^2)|^3 = -8(x + x^2)^3 = 0$, we have $(x + x^2)^3 = 0$ (and $(x - x^2)^3 = 0$) by the above, and therefore $x^5 - x^3 = (x + x^2)^3 - (x + x^2)^{(2)}x^2 = 0$.

(2) By the proof of (1), we obtain $8a = 0$ and $a^3 = 0 = a^{(2)}$ for all $a \in N$. Hence $4a = -2a^2 = a^3 = 0$.

Lemma 5. *Let $x \in R$. If $2x \in N$ and $x^n - x^{n+2^k} \in N$ for some integers $n > 0$ and $k \geq 0$, then $x - x^2 \in N$.*

Proof. As is easily seen,

$$(x - x^2)^{2^k} x^n = (x^n - x^{n+2^k}) x^{2^k} + \sum_{i=1}^{2^k-1} (-1)^i \binom{2^k}{i} x^{2^k-i+n} + 2x^{2^{k+1}+n} \in N.$$

Hence $x - x^2 \in N$.

Lemma 6. *The following are equivalent:*

1) *R is normal.*

2) *If $e, f \in E$ and $e - f \in N^*$, then $e = f$.*

In particular, if $()$ holds in $-E$, then R is normal.*

Proof. If $e, f \in E$, $ef = fe$ and $e - f \in N^*$, then $e - f = (e - f)^3 = 0$. Conversely, suppose 2). Let $e \in E$, and $x \in R$. Then $f = e - ex(1 - e) \in E$ and $e - f = ex(1 - e) \in N^*$. Hence we have $ex = exe$; similarly, $xe = exe$. This proves that R is normal. Now, let $e, f \in E$. Then $(-e + (-e)(-f)) \circ (-f + (-f)(-e)) = ef + fe - e - f$. This enables us to see the latter assertion.

Corollary 1. *Suppose that $x^2y - y^2x \in N \cap C$ for all $x, y \in R \setminus N$. Then $x - x^2 \in N$ for all $x \in R$, and R is normal.*

Proof. If $x \in N$, clearly $x - x^2 \in N$. If $x \in R \setminus N$, then $(x - x^2)x^3 = x^2 \cdot x^2 - (x^2)^2x \in N$. Thus $x - x^2 \in N$ for all $x \in R$. Now, let $e, f \in E$ and $e - f \in N^*$. Then $ef + fe = e + f$ and $ef - fe \in C$, and so $e = e(e + fe - e - f)e + e = efe = efe + |e(e + fe) - (ef - fe)e| = -efe + ef + fe = -e + e + f = f$. Hence R is normal, by Lemma 6.

Lemma 7. *Let R be a ring with 1. If $U \subseteq E + N$, then $2 \in N$. If, furthermore, R is normal and for each $x \in R \setminus U$ there exist integers $n > 0$ and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, then $x - x^2 \in N$ for all $x \in R$.*

Proof. Let $-1 = e + a$, $e \in E$ and $a \in N$. Then $-(1 + a) = e = 1$, since $-(1 + a) \in U$. Hence $2 = -a \in N$. If R is normal, then $u - u^2 \in N$ for any $u \in U$. Now, the latter assertion is clear, by Lemma 5.

We are now ready to complete the proof of our first theorem.

Theorem 1. *The following are equivalent :*

- 1) R satisfies $(*)$.
- 2) R is commutative, $x - x^2 \in N$ for all $x \in R$ (or R/N is a Boolean ring), and $a^{(2)} = 0$ for all $a \in N$.
- 3) R is normal, $x - x^2 \in N$ for all $x \in R$, and $a^{(2)} = 0$ for all $a \in N$.
- 4) R is $(E-N)$ representable and $a^{(2)} = 0$ for all $a \in N$.
- 5) R is normal and satisfies the identity $(X + X^2)^{(2)} = 0$.
- 6) R is normal, N satisfies the identity $(X + X^2)^{(2)} = 0$, and $x - x^2 \in N$ for all $x \in R$.
- 7) R is normal, $2R \subseteq N$, for each $x \in R$ there exist integers $n > 0$ and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, and $a^{(2)} = 0$ for all $a \in N$.
- 8) N satisfies the identity $(X + X^2)^{(2)} = 0$, and $x^2y - y^2x \in N \cap C$ for all $x, y \in R \setminus N$.

Proof. By Lemma 6, 1) implies 5).

3) $\Rightarrow$ 4) $\Rightarrow$ 2). By Lemma 1, and Lemmas 1 and 3, respectively.

5) $\Rightarrow$ 7) $\Rightarrow$ 3). By Lemma 4 (1), and Lemma 5, respectively.

8) $\Rightarrow$ 6) $\Rightarrow$ 3). By Corollary 1, and Lemma 4 (2), respectively.

1) $\Rightarrow$ 8). We have seen that 1) implies 3) and 2). Hence $x^2y - y^2x = (x^2 - x)y - (y^2 - y)x \in N$ for all $x, y \in R$.

2) $\Rightarrow$ 1). Let $x, y \in R$, and put $a = x + xy$, $b = y + yx$. Obviously, $x + x^2 \in N$, and $(x + x^2)^{(2)} = 0$. Conversely, if $a \circ b = 0$ then $a^2 + (a + a^2)b = a(a \circ b) = 0$, and so $a^2 = -(a + a^2)b \in N$. This implies that $a \in N$. Hence $y + xy = 0 \circ b = a^{(2)} \circ b = a \circ (a \circ b) = a \circ 0 = x + xy$, whence $y = x$ follows.

The next includes [4, Theorems 5.5, 5.6 and Corollaries 5.1, 5.3, 5.7] and improves [4, Theorems 5.14, 5.15 and Corollary 5.6].

Corollary 2. *Let R be a ring with 1. Then the following are equivalent :*

- 1) R satisfies $(*)$.
- 2) R is commutative, R/N is a Boolean ring, and $u^2 = 1$ for all $u \in U$ (or $(1 + a)^2 = 1$ for all $a \in N$).
- 3) R is normal, $x - x^2 \in N$ for all $x \in R$, and $u^2 = 1$ for all $u \in U$.
- 4) R is $(E-N)$ representable and $u^2 = 1$ for all $u \in U$.
- 5) R is normal and satisfies the identity $(X + X^2)^{(2)} = 0$.
- 6) R is normal, N satisfies the identity $(X + X^2)^{(2)} = 0$, and $x - x^2 \in N$ for all $x \in R$.
- 7) R is normal, $2 \in N$, and for each $x \in R$ there exist integers $n > 0$

and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, and $u^2 = 1$ for all $u \in U$.

8) N satisfies the identity $(X+X^2)^{(2)} = 0$, and $x^2y - y^2x \in N \cap C$ for all $x, y \in R \setminus N$.

9) R is normal, U satisfies the identity $(X+X^2)^{(2)} = 0$, and $x - x^2 \in N$ for all $x \in R$.

10) R is normal, $2 \in N$, and for each $x \in R$ there exists a positive integer n such that $x^n - x^{n+2} = 0$.

11) R is normal, $2 \in N$, for each $x \in R$ there exist integers $n > 0$ and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, and if $u, v \in U$ and $u - v \in N$ then $u^2 = v^2$.

12) R is normal, $U \subseteq E + N$, for each $x \in R \setminus U$ there exist integers $n > 0$ and $k \geq 0$ such that $x^n - x^{n+2^k} \in N$, and if $u, v \in U$ and $u - v \in N$ then $u^2 = v^2$.

Proof. Obviously, 1) $\Leftrightarrow$ 11) and 12), and the equivalence of 1)–10) is clear by Lemma 4 (1) and Theorem 1.

11) (resp. 12)) $\Leftrightarrow$ 3). By Lemma 5 (resp. Lemma 7), $x - x^2 \in N$ for all $x \in R$. In particular, for each $u \in U$, we obtain $1 - u = u^{-1}(u - u^2) \in N$, and so $1 = u^2$.

Theorem 2. *The following are equivalent :*

1) R satisfies (*).

2) $2R \subseteq N$, and there exists a subset A of R containing $N \cup (-E)$ such that (*) holds in A and $R \setminus A \subseteq E + N$.

3) R is normal, and there exists a subset A of R containing N and satisfying the identity $(X+X^2)^{(2)} = 0$ such that $R \setminus A \subseteq E + N$.

Proof. By Theorem 1, 1) $\Leftrightarrow$ 2) and 3).

2) $\Leftrightarrow$ 1). By Lemma 6, R is normal, and so $x - x^2 \in N$ for all $x \in R \setminus A$. Now, let $x \in A$. Then $(x - x^2)^2 = (x + x^2)^2 - 4x^3 = -2(x + x^2 + 2x^3) \in N$. Hence $x - x^2 \in N$ for all $x \in R$, and therefore R satisfies (*), by Theorem 1 6).

3) $\Leftrightarrow$ 1). In view of Theorem 1, it suffices to show that $x - x^2 \in N$ for all $x \in R$. First, we consider the case that $x \in A$. If $-x \notin A$, clearly $x - x^2 \in N$. If $-x \in A$ then, by the proof of Lemma 4 (1), $8x = 0$, and so $2x \in N$. Hence $(x - x^2)^2 = (x + x^2)^2 - 4x^3 = -2(x + x^2 + 2x^3) \in N$; $x - x^2 \in N$. Next, we consider the case that $x \notin A$: $a = x + x^2 \in N$. Since $2x \notin A$ forces a contradiction $2x = 4a - (2x + 4x^2) \in N \subseteq A$, we see that $2x \in A$. Then $(4a - 2x)^2 = (2x + 4x^2)^2 = -2(2x + 4x^2) = -2(4a$

$-2x$), whence $4(x-x^2) \in N$ follows. Combining this with $x+x^2 = a \in N$, we obtain $8x \in N$, and so $2x^2 \in N$ and $x-x^2 = a-2x^2 \in N$.

Let R be a ring with 1. A subset A of R is called a *weakly normal subset* if for each $x \in R$, either $-x$ or $x-1$ is in A ; a weakly normal subset A of R is called a *normal subset* if $e, f \in E$ and $e-f \in N^*$ imply $-e, -f \in A$ or $-e, -f \notin A$. As is easily seen, if a weakly normal subset A of R satisfies the identity $(X+X^2)^{(2)} = 0$ then R satisfies the same identity; if (*) holds in a normal subset A of R then R is normal. (Note that if $e, f \in E$, then $(-e+(-e)(-f)) \circ (-f+(-f)(-e)) = ef+fe-e-f$ and $(e-1+(e-1)(f-1)) \circ (f-1+(f-1)(e-1)) = ef+fe-e-f$.)

The next includes [4, Theorems 5.1, 5.2, 5.7, 5.12 and 5.13].

Corollary 3. *Let R be a ring with 1. Then the following are equivalent :*

- 1) R satisfies (*).
- 2) $2 \in N$, and there exists a subset A of R containing $N \cup (-E)$ such that (*) holds in A and $R \setminus A \subseteq E+N$.
- 3) There exists a subset A of R containing $U \cup (-E)$ such that (*) holds in A and $R \setminus A \subseteq E+N$.
- 4) R is normal, and there exists a subset A of R containing N and satisfying the identity $(X+X^2)^{(2)} = 0$ such that $R \setminus -A \subseteq E+N$.
- 5) There exists a subset A of R satisfying the identity $(X+X^2)^{(2)} = 0$ such that $A \supseteq N$, $(-A) \cap E \subseteq \{0, 1\}$ and every element in $R \setminus -A$ is uniquely expressible as $e+a$ with $e \in E$ and $a \in N$.
- 6) R is normal, and there exists a weakly normal subset A of R satisfying the identity $(X+X^2)^{(2)} = 0$.
- 7) There exists a normal subset A of R in which (*) holds.

Proof. Obviously, $1) \Rightarrow 2)-7)$. By Theorem 2, each of 2) and 4) implies 1). Further, combining Corollary 2 with the remark stated just above, we readily see that each of 6) and 7) implies 1).

$3) \Rightarrow 1)$. Obviously, $8 = (1+1^2)^2 + 2(1+1^2) = 0$, and so $2 \in N$. Furthermore, R is normal, by Lemma 6. Now, it is easy to see that $x-x^2 \in N$ for all $x \in R$. (See the proof of $2) \Rightarrow 1)$ of Theorem 2.) Hence R satisfies (*), by Corollary 2 9).

$5) \Rightarrow 1)$. By Lemma 6 and Theorem 2.

The next proves the latter part of [2, Theorem 2] and improves [12, Theorem B].

Theorem 3. (1) *If for each $x \in R$ there exists a positive integer n such that $x^{n+1} = x^n$, then N is an ideal of R and R/N is a Boolean ring.*

(2) *Let R be an s -unital ring. If for each $x \in R$ there exists a positive integer n such that $x^{n+1} - x^n \in C$, then R is commutative.*

Proof. (1) In the complete matrix ring $M_t(D)$ over a division ring D with $t > 1$, $(1 + e_{12})^{k+1} \neq (1 + e_{12})^k$ for each positive integer k . Thus, in virtue of the structure theorem of primitive rings, we can easily see that any primitive homomorphic image of R is a division ring. This shows that R/J is a reduced ring, where J is the Jacobson radical of R . Since J is a nil ideal, we conclude that $J = N$ and R/N is a Boolean ring.

(2) In virtue of [6, Proposition 1], we may assume that R has 1. Let x be an arbitrary element in R . Then there exists a positive integer n such that $(1+x)^{n+1} - (1+x)^n \in C$. Since $(1+X)^{n+1} - (1+X)^n = X - X^2 f(X)$ with some $f(X) \in \mathbb{Z}[X]$, R is commutative by [5, Theorem 19].

Now, we shall reprove [2, Theorem 1].

Theorem 4. *The following are equivalent :*

- 1) R is a Boolean ring.
- 2) R is an s -unital, π -regular B' -ring.
- 3) R is an s -unital B' -ring satisfying the identity $(X + X^2)^{(2)} = 0$.
- 4) R is a cs^* -unital B' -ring and an NI-ring.
- 5) R is a B' -ring and an I-ring.
- 6) R is a semiprime I-ring and N^* is commutative.
- 7) R is a semiprime NI-ring and PI ring, and N^* is commutative.
- 8) R is an s -unital ring, and for each $x \in R$ there exists a positive integer n such that $x^{n+1} = x^n$.

Proof. Obviously, 1) implies 3), 4), 7) and 8).

3) $\Leftrightarrow$ 2). By Lemma 4 (1).

4) $\Leftrightarrow$ 5). By [7, Lemma 1].

5) $\Leftrightarrow$ 1). Let $a \in N^*$, and choose $e \in E$ with $eae = a$. Then $e - a = (e - a)^2(e + a)$, and $e - a \in E$, whence $a = 0$ follows. Hence $N = 0$ and E is central.

2) $\Leftrightarrow$ 1). As above, we see that $N = 0$. Now, let $x \in R$. Then there exists $y \in R$ such that $x^n y x^n = x^n$ for some n . Since $x^n y$ and $y x^n$ are central idempotents, we obtain $x^{2n} y = x^n = y x^{2n}$. As is well-known, there exists $z \in R$ such that $xz = zx$ and $x^{n+1} z = x^n$. Then $(x - x^2 z)^n =$

$$\sum_{i=0}^n (-1)^i \binom{n}{i} x^{n+i} z^i = \sum_{i=0}^n (-1)^i \binom{n}{i} x^n = (x-x)^n = 0, \text{ whence } x = x^2 z.$$

This proves that R is strongly regular, and consequently Boolean.

6) $\Rightarrow$ 1). Let $e \in E$. Then $(1-e)ReR(1-e)Re = (1-e)R\{eR(1-e) \cdot (1-e)Re\}e = (1-e)R\{(1-e)Re \cdot eR(1-e)\}e = 0$, whence $(1-e)Re = 0$ follows; similarly, $eR(1-e) = 0$. Hence E is central and R is Boolean.

7) $\Rightarrow$ 6). By [9, Theorem 3], $N = 0$.

8) $\Rightarrow$ 1). By Theorem 3 (1), it suffices to show that $N = 0$. Suppose, to the contrary, that $N \neq 0$, and choose a non-zero a in N^* . Then there exists an idempotent e such that $ea = ae = a$. By hypothesis, there exists a positive integer n such that $(e+a)^{n+1} = (e+a)^n$. But this forces a contradiction $a = 0$.

Corollary 4 (cf. [2, Lemma 1 (3) and Theorem 2]). *If R is a π -regular B' -ring, then for each $x \in R$ there exists a positive integer n such that $x^{n+1} = x^n$.*

Proof. There exists $y \in R$ such that $x^m y x^m = x^m$ for some m . Then $e' = x^m y$ is an idempotent and $e' R e'$ is a Boolean ring by Theorem 4 2). Hence $x^{2m} y = e' x^m e'$ is an idempotent, and so $x^{2m} = x^{2m} y x^m = (x^{2m} y)^2 x^m = x^{3m}$. This proves that $e = x^{2m}$ is in E . Again by Theorem 4 2), $e R e$ is a Boolean ring, and therefore $x^{2m+2} = e x^2 = (e x e)^2 = e x e = x^{2m+1}$.

Finally, we state the following which includes [1, Theorems 1, 2 and 3]

Theorem 5. *Let R be an NI-ring.*

(1) *If R is Artinian, then N is a nilpotent ideal of R and R/N is the finite direct sum of copies of $\text{GF}(2)$.*

(2) *If R is a π -regular PI ring, then N coincides with the prime radical of R and R/N is a Boolean ring.*

(3) *If N is commutative, then N is a commutative ideal of R and R/N is a Boolean ring.*

Proof. (1) As is well-known, the Jacobson radical J of R is nilpotent and R/J is a finite direct sum of matrix rings over division rings. Then, by [7, Lemma 1], R/J is a Boolean ring and $J = N$.

(2) This is [7, Corollary 1].

(3) By [3, Theorem 2], N is a commutative ideal of R . Since R/N is

a reduced I -ring, it is normal and Boolean.

REFERENCES

- [1] H. ABU-KHUZAM: A note on rings which are multiplicatively generated by idempotents and nilpotents, *Internat. J. Math. Math. Sci.* 11 (1988), 5–8.
- [2] H. ABU-KHUZAM and H. TOMINAGA: On rings in which every strongly regular element is idempotent, *Proc. 10th Symp. on Semigroups* (1986), 61–64.
- [3] H. E. BELL and H. TOMINAGA: On periodic rings and related rings, *Math. J. Okayama Univ.* 28 (1986), 101–103.
- [4] J. GROSEN: Rings satisfying polynomial identities or constraints on certain subsets, Thesis, University of California, Santa Barbara, 1988.
- [5] I. N. HERSTEIN: The structure of a certain class of rings, *Amer. J. Math.* 75 (1953), 864–871.
- [6] Y. HIRANO, Y. KOBAYASHI and H. TOMINAGA: Some polynomial identities and commutativity of s -unital rings, *Math. J. Okayama Univ.* 24 (1982), 7–13.
- [7] Y. HIRANO and H. KOMATSU: A characterization of Boolean rings (III), *Math. J. Okayama Univ.* 27 (1985), 33–34.
- [8] Y. HIRANO, H. KOMATSU, H. TOMINAGA and A. YAQUB: On rings satisfying the identity $X^{2k} = X^k$, *Math. J. Okayama Univ.* 30 (1988), 25–31.
- [9] Y. HIRANO and H. TOMINAGA: Some remarks on the set of idempotents and the set of elements square zero, *Math. J. Okayama Univ.* 25 (1983), 133–138.
- [10] Y. HIRANO, H. TOMINAGA and A. YAQUB: On rings in which every element is uniquely expressible as a sum of nilpotent element and a certain potent element, *Math. J. Okayama Univ.* 30 (1988), 33–40.
- [11] H. KOMATSU and H. TOMINAGA: On s' -unital, s^* -unital, and s^{**} -unital rings, *Proc. 8th Sympos. on Semigroups* (1984), 45–51.
- [12] M. A. QUADRI and M. ASHRAF: Commutativity of generalized Boolean rings, *Publ. Math. Debrecen* 35 (1988), 73–75.

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